Investigation of motions in a central force field using the Dynamics Solver program – computer-aided experimental physics

1. Introduction

It is common experience that students' level of knowledge, skills, and motivation in the field of science have all declined sharply in recent years. The reasons for this are manifold: on the one hand, there are fewer young people due to demographic decline; on the other hand, the standard of secondary education has declined noticeably; finally, the impact of the whole society turning away from natural sciences cannot be neglected either. It has become clear that a kind of paradigm shift is needed in science education, instead of the descriptive-explanatory methods an illustration-oriented approach should be applied, which can make physics education more enjoyable and successful. During curriculum development, the exact, field-specific sub-results should be simplified in such a way that the essence is retained, at the same time the details should be moulded into a whole with a unified approach and arousing interest. An important aspect is the inclusion of new, preferably practice-oriented topics in education, as these can increase students' interest and show the extent to which physics plays a decisive role in everyday life.

Computers have opened up a new dimension for physics as well, with the emergence of computer-aided experimental physics as a completely new method of study. With the help of computer simulations, we can obtain relevant quantitative information about models that were previously not discussed at all or only qualitatively. Dynamics Solver (hereinafter DS) is a freely downloadable program [1], which is specifically designed to simulate dynamic systems. For the authors of the present study, its use has brought real breakthroughs in both education and research.

The essential features of Dynamics Solver:

- freely downloadable,
- requires minimum programming knowledge to use,
- characterised by high level of validity, strong reliability,
- extremely fast,
- amazingly flexible, almost any dynamic system model can be specified in it.

Using DS does not assume any previous programming knowledge: all the information needed for the simulation is entered through user-friendly dialog boxes and the display and extraction of a wide range of graphical and numerical results is very simple. The program's powerful built-in compiler turns a wide range of standard-form mathematical expressions into outstandingly fast-running internal code. Because of the above, DS is a highly effective tool for studying dynamic systems.

Models created in Dynamics Solver are saved as ASCII text *problem files* with **.ds* extension by the program, so one could actually write them with a simple text editor, but of course the creation of the model is much more obvious and clear in the very user-friendly interface of the program. We created a brief overview for using the program in Hungarian [2] and English [3], but the most basic functions required to run the **.ds* problem files related to this study are also listed here briefly: the  icon is used for running, the  icon is used for stopping (pausing), the  icon is used for continuing, the  icon is used for deleting the graphical windows, the  icon is used for displaying the parameter table and the  icon is used for displaying the initial conditions table.

With the help of DS, students can understand the basic concepts and methods of dynamics almost by playing, and by using simulations they can experience the feeling of research and discovery.

2. The central force field and the emerging forms of motion

In the case of a so-called **central force field**, the potential energy determining the motion of an arbitrary point of mass depends only on the distance *r* measured from a given point (centre), so:

$$V(r). \qquad (2.1.a)$$

In this case the force acting on the point of mass is

$$\vec{F} = -\frac{dV}{dr}\frac{\vec{r}}{r}, \qquad (2.1.b)$$

whose line of action always passes through a fixed point of the reference frame, the centre (in most cases we choose the origin ***O*** of our coordinate system here) (see Figure 1).

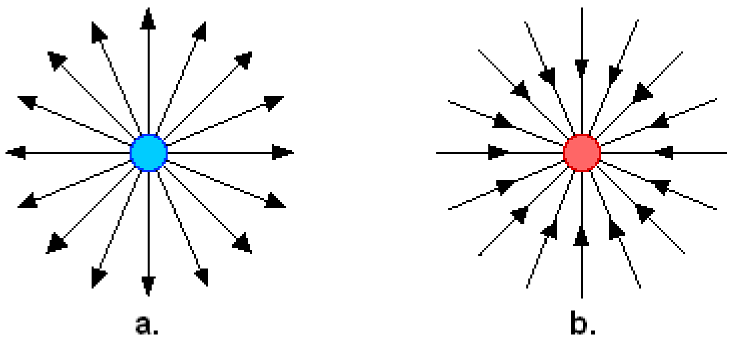

Figure 1. Representing a central force field (a.: repulsive, b.: attractive force field)

The most important central force fields - which are also the most common in nature - are those in which potential energy is inversely proportional to distance *r*:

$$V(r) = -\frac{\alpha}{r}, \qquad (2.2.a)$$

where $\alpha > 0$ for an attractive and $\alpha < 0$ for a repulsive force field. From (2.1.b) force is proportional to $\frac{1}{r^2}$, where *r* is the distance:

$$F(r) = \frac{\alpha}{r^2}. \qquad (2.2.b)$$

The two most important, most well-known $1/r$ type central force fields are:

- gravitational force (the force field of a body with mass *M*, under the condition $M >> m$): $\alpha = G \cdot M \cdot m$,
- Coulomb force (the force field of a body with mass *M* and charge *Q*, under the condition $M >> m$): $\alpha = \frac{-Q \cdot q}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r}$.

A brief theoretical discussion of motions resulting in the central force fields given by formulas (2.2) is given in the Appendix. In this section, we only highlight the most important results derived for the forms of motion. Essentially, the trajectory of the resulting motion is the conic section with equation:

$$r(\varphi) = \frac{p}{1 + e \cdot \cos\varphi} \qquad (2.3.a)$$

(in polar coordinate system), whose parameters are

$$p = \frac{J^2}{m\alpha} \text{ and } e = \sqrt{1 + \frac{2EJ^2}{m\alpha^2}} \text{ , (2.3.b)}$$

where E is the total mechanical energy of the body and J is the angular momentum (rotational momentum) defined by formula $\vec{J} = m \cdot \vec{r} \times \vec{v}$.

We have also concluded that the trajectory of a moving body of mass m will be closed (finite) or open (infinite) depending on the total mechanical energy E of the body.

2.1. Closed trajectories (orbits)

In the case of $E < 0$ the body with mass m moves on an *elliptical trajectory with parameter p and eccentricity e*, for which:

- the major and minor axes of the ellipse are $a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}$, $b = a \cdot \sqrt{1-e^2}$, (2.4.a)
- and the orbital period is $T_p = \pi\alpha\sqrt{\frac{m}{2|E|^3}}$. (2.4.b)

(a) Gravitational field (planetary motion, Kepler's laws)

In the gravitational force field of a body with mass M ($\alpha = G \cdot M \cdot m$), using formulas (2.4):

$$\frac{a^3}{T_p^2} = \frac{\left(\frac{\alpha}{2|E|}\right)^3}{\left(\pi\alpha\sqrt{\frac{m}{2|E|^3}}\right)^2} = \frac{G \cdot M}{(2\pi)^2} = constant \qquad (2.5)$$

which is **Kepler's 3rd law**, and the appearance of the elliptical trajectory itself is **Kepler's 1st law**. In Appendix F.1 we show that in a central force field angular momentum is constant, so *the area swept out by a point of mass moving in a central force field in a unit time,*

$$\dot{T} = \frac{1}{2}\left|v_x \cdot y - v_y \cdot x\right| \qquad (2.6)$$

is constant. This is **Kepler's 2nd law**: the line joining a planet with the Sun sweeps out equal areas in equal time intervals.

(Remark: in the case of $e = 0$, that is, in the case of an energy value $E = E_{circle} = -\frac{m\alpha^2}{2J^2}$ $a = b = p$, so the trajectory is a circle.

(b) Electrostatic field (electron orbits, Bohr-Sommerfeld model)

In the quasi classical quantum theory, according to the Bohr-Sommerfeld model describing the structure of atoms, negatively charged electrons are located in elliptical orbits around a positively charged nucleus, so they essentially form a tiny solar system. In this case, instead of gravitational force electrostatic Coulomb force acts (attractive in the case of opposite charges) and

$$\alpha = \frac{-Q \cdot q}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r}.$$

The full physical state of all electrons in an atom can be described by four quantum numbers.

The *principal quantum number n* (*n* = 1, 2, 3, …) describes the size (the average distance of the electron from the nucleus) and the energy of the orbit, while the *azimuthal quantum number l* (l = 0, 1, 2, 3, …, n-1) describes the shape and the angular momentum of the orbit (see Figure 2). We now know that this picture is not realistic, but within a given framework it is very successful (e.g. in the description of chemical properties) and, above all, very illustrative.

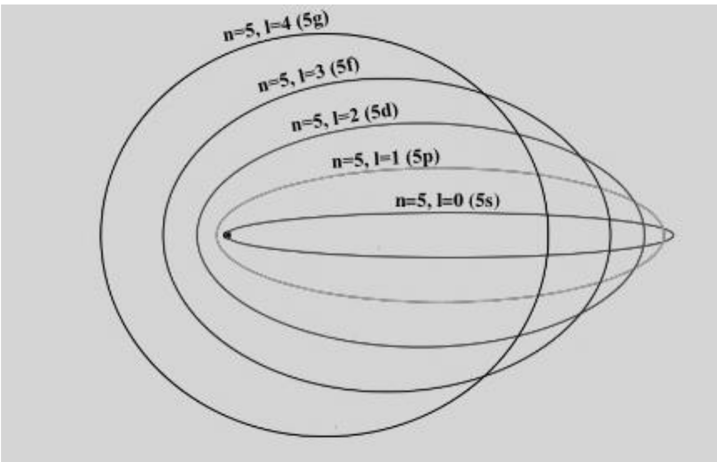

Figure 2. Electron orbits in the Bohr-Sommerfeld atom model

2.2. Open orbits

In the case of $E > 0$ *the body moves on an open (infinite) hyperbolic trajectory* (in the special case of $E = 0$ the trajectory is a parabola).

A more detailed discussion is given in Appendix F.2, here we summarize only the most important practical examples.

2.2.a. Deflection in a central gravitational force field

Figure 3 shows the hyperbolic trajectory of a body moving in an attractive central force field, the body with mass *m* moving at an initial speed v_0 at a large distance from the centre of attraction would pass at distance *D* from the object with mass *M* ($M >> m$), if it was not deflected by the gravitational field of the centre of attraction.

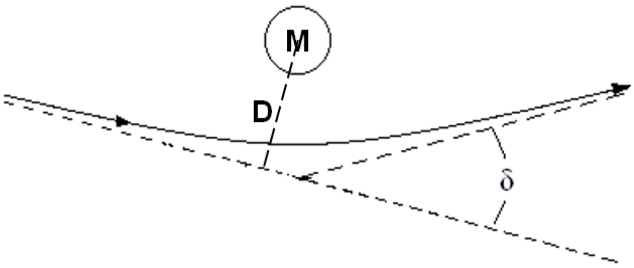

Figure 3. Deflection of trajectory in a central gravitational force field

Thus, in the gravitational force field of mass *M*, it is deflected by angle δ moving in a hyperbolic orbit. The hyperbola has two non-intersecting and non-touching arms, the trajectory of the body being the hyperbola arm closer to the centre of attraction. As the distance from the axis of symmetry increases beyond all limits, the two ends of the hyperbola arm approach two straight lines called asymptotes, the deflection δ is the angle enclosed by the two asymptotes.

According to the formula (F.2.8) obtained through purely classical physical derivation

$$\delta = 2\frac{G \cdot M}{D \cdot v_0^2} \quad . \qquad (2.7)$$

2.2.b. Particle scattering

The scattering of an electrically charged particle on an electromagnetic force centre or on another charged particle is called Coulomb scattering. It was first investigated experimentally by Rutherford by scattering alpha particles on gold nuclei, called *Rutherford's scattering experiment* (Figure 4).

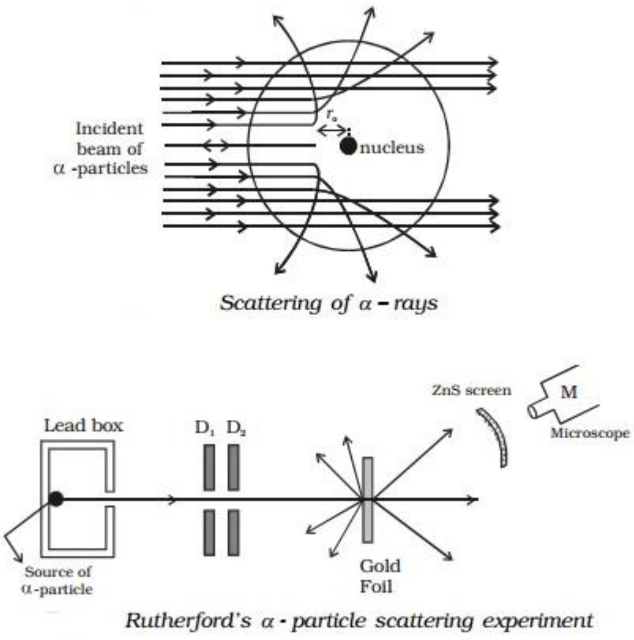

Figure 4. Rutherford scattering

2.2.c. Deflection of light

In modern physics (quantum theory and relativity), light is also considered to be a material object with mass, so perhaps the most important application of our considerations – also of outstanding history of science importance - is the deflection of light when passing near a massive body (e.g. the Sun). Unfortunately, classical physics becomes inaccurate at high speeds, so in this case we have to rely on the theory of relativity, which results in a relativistic angular deflection being twice the value given by the classical formula (see Appendix F.2):

$$\delta = 4\frac{G \cdot M}{D \cdot v_0^2} \quad . \qquad (2.8)$$

The deflection of light close to the Sun provided the first experimental evidence to prove general relativity. In 1919, utilizing a solar eclipse, English astronomer A. Eddington measured the deflection of light from stars near the Sun (while moving towards the Earth their light passes close to the Sun) and the results of the measurement clearly matched the relativistic formula (see Table 1).

	deflection of light close to the Sun (arcseconds)
calculated based on classical physics	0.87
calculated based on the theory of relativity	1.75
measured (Eddington)	1.74±0.03

Table 1. Deflection of light close to the Sun

3. Simulations – computer-aided experimental physics

The exercises discussed in this chapter offer the experience of real *computer-aided experimental physics*, the joy of personal "discovery", so it is worth experimenting with your own attempts, it is worth the time dedicated to it. Therefore, we recommend that you try to find your own solutions to the following exercises, but we provide some help in Appendix F.3.

Remark: the problem of central force field also can be simulated in Excel, see for example on page *http://theorphys.elte.hu/fiztan/num/* using Euler method..

In the following exercises, study the numerical simulation of motions resulting in a gravitational force field with $\alpha = G \cdot M \cdot m$ by running problem file *grav_sim.ds*, which can be found in folder *DS*.

If you would like to understand the operation of the simulation in detail, then check the structure of the problem file (following the help mentioned in the Introduction through menu items *Type…, Variables…, Parameters…, Equations…, Initial conditions… and Range…* in the *Edit* menu). This is absolutely not necessary; you can experiment with the simulation program using the basic functions described at the end of the Introduction.

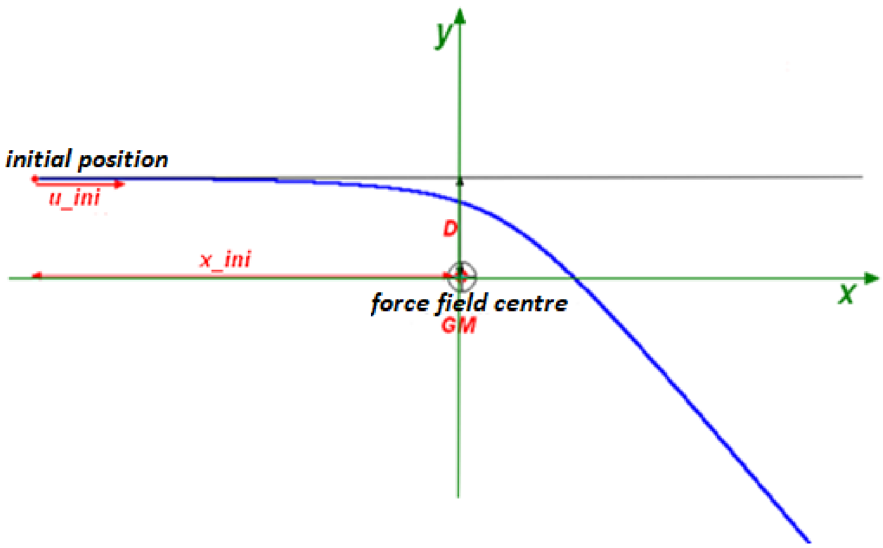

Figure 5. The coordinate system and parameters used in the simulation

By placing the origin of the (Cartesian) coordinate system in the centre of the force field, **four parameters determine the motion**:

- parameter ***GM*** determining the strength of the force field (which in the case of gravitational force field is $GM = G \cdot M$),
- and the initial conditions, that is, the initial position $\left(x_0 = x\_ini \ ; \ y_0 = D\right)$ given by parameters ***x_ini*** and ***D***, and the initial velocity vector $\left(v_{x0} = u_0 = u\_ini \ ; \ v_{y0} = v_0 = 0\right)$ – that is, parallel to the *x* axis – given by parameter ***u_ini*** (the general nature of the motion is not limited by taking the initial velocity to be parallel to the *x*-axis).

We defined a graphical window and a text data display window as output. In the graphical window, the trajectory is drawn in the x-y plane. In the text window, the directional angle of the instantaneous velocity (the angle enclosed by the velocity vector and the *x*-axis) and the magnitude of the area swept out by the position vector pointing from the origin to the current position per unit time are displayed (see Appendix F.1).

Exercise 1.

Perform *computer-aided experimental physics*: by changing the value of the above four parameters try to find a hyperbolic trajectory ("comet") and an elliptical trajectory ("planet"). (If required, change the settings of the graphical display window in the dialog window in menu item *Graphics format…* in the *Output* menu as described in the guide.)

What do you experience if you choose a negative value for the *GM* parameter, so you have a "repulsive" central force field?

Exercise 2.

If you could find a hyperbolic trajectory, then you can see in the text window that the bearing angle starts at 0 radians, which corresponds to the initial velocity in the *x* direction chosen by us (this is actually the starting asymptote of the hyperbolic trajectory), and after a sufficiently long time the bearing angle converges to a non-zero value (this is the direction of the other asymptote of the hyperbolic trajectory). This bearing angle is equal to the angle of deflection δ determined in the theoretical description.

Experiment with the simulation to determine how the angle of deflection δ depends on the parameters. Always change only one of parameters *GM*, *D* and *u_ini* systematically (the *x_ini* parameter is not relevant, but it should be chosen large enough to have the starting point on the input asymptote) to show that the angle of deflection δ is directly proportional to *GM*, inversely proportional to *D* and the square of *u_ini*, so prove formula (2.7) experimentally. (Of course, the classical physical approach is valid, since in the simulation the equations of motion (F.2.3) based on Newton's 2nd axiom are used.)

Exercise 3.

In Exercise 1 you found that in the case of the (attractive) central force field, elliptical trajectories can appear at appropriate parameter values. By placing the Sun in the centre of the force field, you can essentially simulate the motion of the planets! 400 years ago, Johannes Kepler studied the motion of the planets of the Solar System with the help of a fantastic new invention, the telescope, and with tremendous work, systematic and accurate data collection, he condensed his observations into three wonderful laws:

I. The orbit of a planet is an ellipse with the Sun at one of the foci.

II. The line segment joining a planet and the Sun sweeps out equal areas in equal time intervals.

III. The square of a planet's orbital period (T) is proportional to the cube of the semi-major axis (a) of the elliptical orbit, that is:

$$\frac{a^3}{T^2} = constant \qquad \text{(More accurately: } \frac{a^3}{T^2} = \frac{G \cdot M}{(2\pi)^2}\text{)}$$

(The parameters are defined by formulas (2.4))

Kepler's laws are also discussed theoretically in Chapter 2 and in the Appendix, now study the motions formed in closed orbits by computer simulation, so prove Kepler's laws given above experimentally.

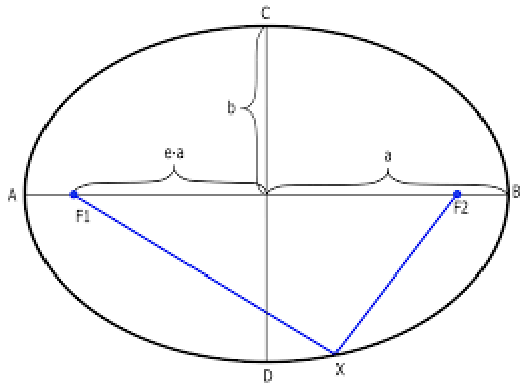

Figure 6. Parameters of the ellipse of a planetary motion

Exercise 4.

In this exercise study the numerical simulation of motions created in the $\alpha = \frac{-Q \cdot q}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r}$ Coulomb's electrostatic force field by running problem file *Rutherford_sim.ds* located in folder *DS*. The coordinate system and the parameters used are the same as in simulation *grav_sim.ds*, the only differences are that this time the strength of the centre is given by parameter α instead of *GM* and that we do not start a single point of mass from a single initial position with distance *x_ini* but a beam of 200 particles spread out uniformly in the *y*-interval between $-D$ and $+D$ similarly to Rutherford's scattering experiment.

Change the parameters and observe the nature of the particle scattering.

Appendix

F.1. Angular momentum in the central force field

The *vector product* (*cross product*) of vectors $\vec{r}\left(x,y,z\right)$ and $\vec{v}\left(v_x,v_y,v_z\right)$ given with Cartesian base vectors (right-twist orthogonal unit vectors) $\left(\vec{i},\vec{j},\vec{k}\right)$ in three dimensions is defined as

$$\vec{r}\times\vec{v}=\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ v_x & v_y & v_z\end{pmatrix}=\vec{i}\left(y\cdot v_z-z\cdot v_y\right)+\vec{j}\left(z\cdot v_x-x\cdot v_z\right)+\vec{k}\left(x\cdot v_y-y\cdot v_x\right) \qquad \text{(F.1.1)}$$

The geometric interpretation of the vector product: the area of the parallelogram defined by the two vectors is equal to the magnitude of their vector product (Figure 7), so $T=\left|\vec{r}\times\vec{v}\right|=\left|\vec{r}\right|\cdot\left|\vec{v}\right|\cdot\sin\varphi$. It follows from the definition, but clearly from the latter interpretation, that the vector product of parallel vectors $\left(\sin\varphi=0\right)$ is zero.

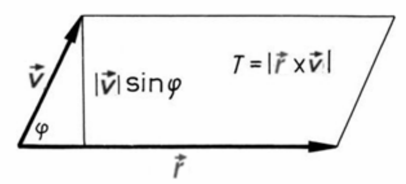

Figure 7. The geometric interpretation of the vector product

Vector product has several important uses, a physical application is discussed below. In classical physics, *angular momentum (rotational momentum)* is a vector quantity characterising the state of rotation of a body. The angular momentum of a point mass with mass *m*, instantaneous velocity $\vec{v}$ for a given point is

$$\vec{J}=m\cdot\vec{r}\times\vec{v}. \quad \text{(F.1.2.)}$$

Let us investigate the temporal change of the angular momentum, that is, its derivative with respect to time, which according to the differentiation rule of the product function is

$$\frac{d}{dt}\vec{J}=m\cdot\frac{d}{dt}\left(\vec{r}\times\vec{v}\right)=m\cdot\left(\frac{d\vec{r}}{dt}\times\vec{v}+\vec{r}\times\frac{d\vec{v}}{dt}\right),$$

as by definition $\frac{d\vec{r}}{dt}=\vec{v}$, $\frac{d\vec{v}}{dt}=\vec{a}$:

$$\frac{d}{dt}\vec{J}=m\cdot\frac{d}{dt}\left(\vec{r}\times\vec{v}\right)=m\cdot\left(\vec{v}\times\vec{v}+\vec{r}\times\vec{a}\right).$$

As the vector product of parallel vectors is zero, the first term in the parenthesis above is zero, since $\vec{v}$ is obviously parallel to itself, and in a central force field force is parallel to the position vector $\vec{r}$ (see (2.1.b)), so the instantaneous acceleration $\vec{a}$ of the body with mass *m* is also parallel to position vector $\vec{r}$, so:

$$\frac{d}{dt}\vec{J}=0\text{, that is, }\vec{J}=constant.$$

Thus, we have obtained that *the angular momentum of a point mass moving in a central force field is constant*!

Now, let us consider the area swept out by a point mass moving in a central force field per unit time, $\dot{\vec{T}}$ (e.g., a planet orbiting the Sun) (see Figure 8). If one of the vectors – in the angular momentum velocity $\vec{v}$ – stands for the displacement in a unit time, then vector product $\vec{r} \times \vec{v}$ gives the area in a unit time, so the magnitude of the area "swept" by the position vector $\vec{r}$ in a unit time is given by the area of the triangle determined by the two vectors (see the figure), which is half of the area of the parallelogram determined by the two vectors, so $\dot{T} = \frac{1}{2}\vec{r} \times \vec{v}$ and its unit is m^2/s.

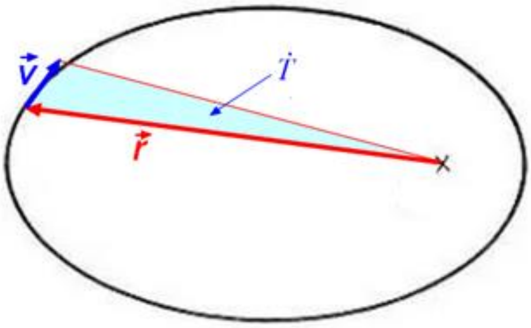

Figure 8. The area swept out by the position vector of a point mass moving in a central force field per unit time

Based on the above formulas, the swept area can be written using the angular momentum:

$$\dot{\vec{T}} = \frac{1}{2}\vec{r} \times \vec{v} = \frac{1}{2m}\vec{J}.$$

As we have shown that in a central force field angular momentum is constant, we have come to the conclusion that *the area swept out by a point of mass moving in the central force field per unit time,* $\dot{\vec{T}}$ *is constant*. (This is **Kepler's 2nd law**: The line segment joining a planet and the Sun sweeps out equal areas in equal time intervals.)

If the plane of the orbit is in the x-y plane of the coordinate system ($z = 0$), then $\vec{r}(x, y, 0)$ and $\vec{v}(v_x, v_y, 0)$, so their vector product is

$$\vec{r} \times \vec{v} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & 0 \\ v_x & v_y & 0 \end{pmatrix} = \vec{i}\left(y \cdot 0 - 0 \cdot v_y\right) + \vec{j}\left(0 \cdot v_x - x \cdot 0\right) + \vec{k}\left(x \cdot v_y - y \cdot v_x\right) = \vec{k}\left(x \cdot v_y - y \cdot v_x\right),$$

so the magnitude of the area swept out per unit time is

$$\dot{T} = \frac{1}{2}\left|v_x \cdot y - v_y \cdot x\right|. \quad \text{(F.2.3.)}$$

F.2. Motion of a point of mass in a central force field with potential 1/r

Let us consider a body with mass *m* moving in the central force field of a large body with mass *M* (*M* >> *m*), which is characterised by potential

$$V(r) = -\frac{\alpha}{r} \qquad \text{(F.2.1.)}$$

(if $\alpha > 0$, the field is attractive, if $\alpha < 0$, the field is repulsive).

Then the force is

$$\vec{F} = -\frac{dV}{dr}\frac{\vec{r}}{r} = -\frac{\alpha}{r^2}\frac{\vec{r}}{r}, \qquad \text{(F.2.2.)}$$

Thus, according to Newton's 2nd axiom the equation of motion of a body with mass *m* is

$$\vec{F} = -\frac{\alpha}{r^2}\frac{\vec{r}}{r} = m \cdot \vec{a},$$

from which the acceleration vector in the x-y plane of the trajectory is

$$\vec{a}\left(a_x = -\frac{\alpha x}{mr^3}; a_y = -\frac{\alpha y}{mr^3}\right), \text{ where } r = \sqrt{x^2 + y^2} \qquad \text{(F.2.3.)}$$

A detailed discussion can be found in Sections 14. and 15. of [4], here only the essential steps are reviewed.

It is useful to introduce the so-called *effective potential*

$$V_{eff}(r) = -\frac{\alpha}{r} + \frac{J^2}{2mr^2} \qquad \text{(F.2.4.)}$$

(see Figure 9.), where *J* is the angular momentum discussed in the previous section.

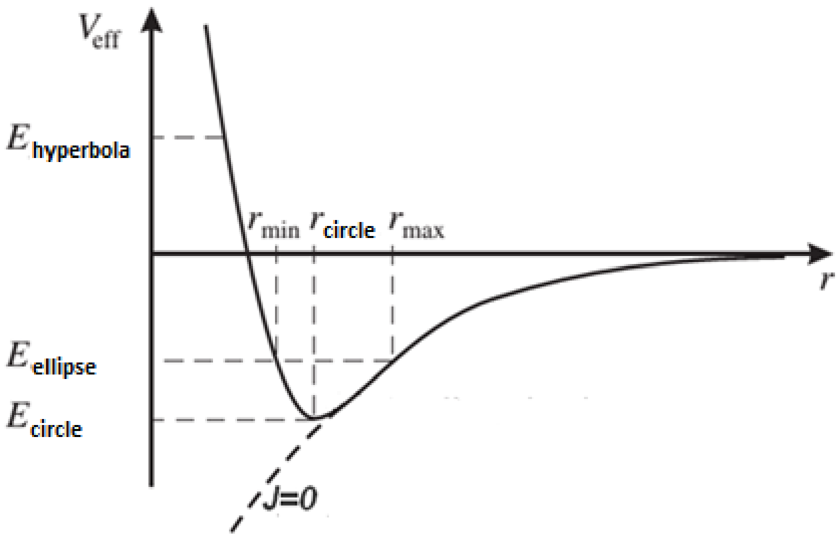

Figure 9. The effective potential as a function of distance

Switching from the x-y Cartesian coordinate system to the r-φ polar coordinate system and performing the elementary integral in the equation of motion, we get that:

$$r(\varphi) = \frac{p}{1 + e \cdot \cos\varphi}, \qquad \text{(F.2.5.a)}$$

where

$$p=\frac{J^2}{m\alpha} \text{ and } e=\sqrt{1+\frac{2EJ^2}{m\alpha^2}}. \quad \text{(F.2.5.b)}$$

(F.2.5.a) is the equation of a *conic section* whose *focus* is at the origin, p is the *parameter* of the trajectory and e is its *eccentricity*.

It depends on the total mechanical energy $E=V_{eff}(r)+\frac{1}{2}mv^2$ of the moving body with mass m whether the orbit is closed (finite) or open (infinite) (see the figure above). As the kinetic energy, $\frac{1}{2}mv^2$ is trivially non-negative, motion can occur only in the range r where $E \geq V_{eff}(r)$ is true.

First consider the case when $E<0$,

- *then the body moves in a (closed) elliptical orbit (***Kepler's 1st law***)*,
- the major and minor axes of the ellipse are: $a=\frac{p}{1-e^2}=\frac{\alpha}{2|E|}$, $b=a\cdot\sqrt{1-e^2}$, (F.2.6.a),
- the orbital period is $T=\pi\alpha\sqrt{\frac{m}{2|E|^3}}$ (F.2.6.b).
- based on (F.2.6.a) and (F.2.6.b) $\frac{a^3}{T^2}=\frac{\alpha}{(2\pi)^2 m}=constant$ (**Kepler's 3rd law**) (F.2.6.c)
- (in the case e = 0, that is, energy $E=E_{circle}=-\frac{m\alpha^2}{2J^2}$, $a = b = p$, so the orbit is a circle).
- (Reminder: **Kepler's 2nd law** was derived in Appendix F.1.)

In the case of $E>0$ *the body moves on an infinite (open) hyperbolic path* (in the case of $E=0$ the orbit is a parabola).

Let us investigate the case $E > 0$ in more detail, when a body of mass m with velocity v_0 (this is the "initial" velocity when it is still very far from the centre of force) would move along a straight line at distance D from an object of mass M ($M >> m$) (if it was not affected by gravitation). In the gravitational force field of mass M, the body moves in a hyperbolic trajectory and deflects through angle δ. The hyperbola has two non-intersecting and non-touching arms, the trajectory of the body is the hyperbola arm closer to the centre of attraction. As the distance from the axis of symmetry increases beyond all limits, the two ends of the hyperbola arms approach two lines called asymptotes, the deflection δ is the angle enclosed by the two asymptotes. Let us determine angle δ.

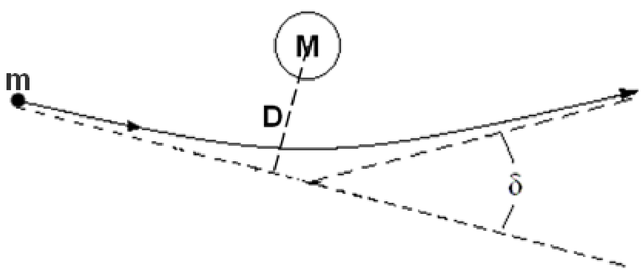

Figure 10. Deflection of a body moving in a hyperbolic trajectory in an attractive central force field of nature 1/r

Let us consider the factors affecting the deflection, first simply on the basis of units of measurement (dimensional considerations), then based on classical physical description, and finally based on the theory of relativity.

a. Dimensional consideration:

Let us take into account the physical factors that could affect deflection δ:

- mass M of the object being the source of the central force field (its SI unit is kg),
- the (Cavendish) gravitational constant G (its SI unit is $\frac{m^3}{kg \cdot s^2}$),
- velocity v_0 (its SI unit is $\frac{m}{s}$),
- distance D (its SI unit is m).

Let us mix the units of the above four quantities to obtain a dimensionless (radian) δ plane angle unit; we conclude very quickly that

$$\delta \propto \frac{G \cdot M}{D \cdot v_0^2} \qquad \text{(F.2.7.)}$$

Of course, the reciprocal of the above expression would also be suitable dimensionally, but it would contradict basic physical requirements, e.g. that the deflection should be proportional to the strength of the centre (the product of mass M and the gravitational constant G) and inversely proportional to the distance D and the initial velocity v_0, which corresponds to our image of the process.

b. Classical physics:

Using classical physical calculation, for a point mass with mass m moving in a central force field with potential (2.2.a) ($\alpha > 0$ attractive, $\alpha < 0$ repulsive nature) at velocity v_0 (at a great distance from the centre) and distance D from the centre, (through derivation not detailed here, see e.g. [4] I. Sections 14, 15, 18 and 19, based on formula (19,1)) we get that the deflection δ is

$$ctg \frac{\delta}{2} = \frac{m \cdot v_0^2}{\left(\alpha / D \right)}$$

The approximation polynomial of the cotangent function (Taylor series to first order) is

$$ctg\ x \cong \frac{1}{x} - \ldots, \text{ so}$$

$$ctg \frac{\delta}{2} \cong \frac{1}{\frac{\delta}{2}} = \frac{2}{\delta} = \frac{m \cdot v_0^2}{\left(\alpha / D \right)}$$

Reorganising and using that fact that for gravitational potential $\alpha = G \cdot m \cdot M$ we obtain that

$$\delta = 2 \frac{G \cdot M}{D \cdot v_0^2} \qquad \text{(F.2.8.)}$$

So according to classical (Newtonian) physics, the angular deflection is really proportional to the expression obtained through dimensional analysis, exactly twice that!

b. Theory of relativity:

Special relativity is based on two postulates. One is the principle of relativity, which states that natural processes happen in the same way when observed from any inertial reference frame and the form of the laws describing them is the same in any two inertial reference frames. The other postulate makes the surprising statement that the speed of light in vacuum is the same for any observer, $c_0 = 3 \cdot 10^8 \ {}^{m}\!/_{s}$. The value of a given physical quantity measured in one or another reference frame may be different, but these can be determined from each other clearly using the so-called *Lorentz transformation*, which characterises the relative motion of the reference frames, there is no separate absolute space and time, Lorentz transformation is essentially a geometric transformation in the 4-dimensional space-time.

The general theory of relativity merges special relativity with Newton's universal law of gravitation, describing gravitation as a geometric property of space-time. The general theory of relativity is based on the *principle of equivalence*, which states that a local gravitational effect corresponds to the effect of an acceleration observed in a gravitation-free spatial reference frame accelerating in space, and (also locally) the two cannot be distinguished. This is not a priori truth, but a statement based on empirical observations (e.g., Eötvös pendulum) to verify the equivalence of inertial and gravitational mass.

Based on calculations not detailed here (see e.g. [5] [6]) we obtain that in a local (non-inertial) reference frame at distance *r* from a body with mass *M*, the speed of light is

$$c(r) = c_0 \sqrt{\frac{1 - \frac{2GM}{c_0^2 r}}{1 + \frac{2GM}{c_0^2 r}}} . \qquad \text{(F.2.9.)}$$

From here, we can move on with completely classical physical considerations. Let us use the (absolute) refractive index $n = \frac{c_0}{c}$ known from optics in a medium where the speed of propagation of light is *c* ($c \le c_0$), which now, based on (F.2.9.) is:

$$n(r) = \frac{c_0}{c(r)} = \sqrt{\frac{1 + \frac{2GM}{c_0^2 r}}{1 - \frac{2GM}{c_0^2 r}}} \cong 1 + \frac{2GM}{c_0^2 r} = 1 + \frac{b}{r} \text{ , where } b = \frac{2GM}{c_0^2} , \qquad \text{(F.2.10.)}$$

where the first order of the Taylor series is used for approximation. (F.2.10.) essentially means a beam of light travelling in an inhomogeneous optical medium. Such problem is widely discussed in classical optics [7] (e.g. in the case of the mirage phenomenon), the resulting angular deflection is

$$\delta = \frac{2b}{D} ,$$

which in this case, using (F.2.10.) is

$$\delta = 4\frac{G \cdot M}{D \cdot v_0^2} \qquad \text{(F.2.11.).}$$

It is not the same as formula (F.2.8.), which was derived using purely classical physics: *the relativistic result is exactly twice the angular deflection obtained from classical physics.*

F.3. Help for the exercises set

If you feel that you do not succeed in solving the exercises set on your own, some help is provided below.

Exercise 1.

A hyperbolic trajectory ("deflection", "comet") appears for example with parameter values *GM* = 20 , *u_ini* = 4 , *x_ini* = -20 , *D* = 3.

An elliptical trajectory ("planet") appears for example with the parameter values below (trajectories are shown in Figure 11.):

- *GM* = 20 , *u_ini* = 2 , *x_ini* = -5 , *D* = 3 (blue, elongated),
- *GM* = 20 , *u_ini* = 3 , *x_ini* = -1 , *D* = 3 (black, less elongated),
- *GM* = 20 , *u_ini* = 2 , *x_ini* = -2 , *D* = 6 (green, almost circular),
- *GM* = 20 , *u_ini* = 2 , *x_ini* = 0 , *D* = 5 (red, circular),

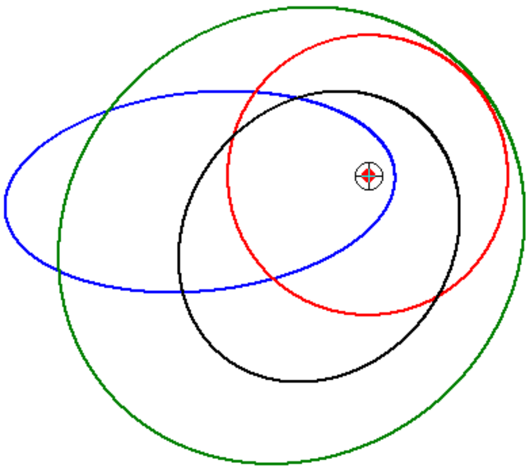

Figure 11. Elliptical trajectories in the DS simulation

If you choose a negative value for the *GM* parameter, so you have a "repulsive" central force field, then it is clear that only hyperbolic trajectories can appear.

Exercise 2.

Changing only one of parameters *GM, D* and *u_ini* systematically, we record the change in the angle of deflection δ. (parameter *x_ini* is irrelevant, for example we can set the value *x_ini* = -20).

(a) investigating the dependence on mass: (*GM*=... , *u_ini* = 4 , *D* = 3)

GM	angle δ [rad]
1	0.0416
2	0.0833

3	0.1253
4	0.1674
5	0.2097

From the above data it can be concluded that the angle of deflection δ is directly proportional to parameter *GM*.

(b) investigating the dependence on distance: (*GM* = 2 , *u_ini* = 4 , *D* =...)

D	angle δ [rad]
1	0.250104
2	0.125307
3	0.0833402
4	0.0622552

From the above data it can be concluded that the angle of deflection δ is inversely proportional to parameter *D*.

(c) investigating the dependence on initial velocity: (*GM*=1 , *u_ini*=... , *D*=3)

u_ini	angle δ [rad]
1	0.671469
2	0.167432
3	0.0740379
4	0.0415587

From the above data it can be concluded that the angle of deflection δ is inversely proportional to the square of parameter *u_ini*.

Combining the above three findings into a single relationship we obtain that

$$\delta \propto \frac{GM}{D \cdot u\_ini^2},$$

more precisely, we obtain

$$\delta = 2\frac{GM}{D \cdot u\_ini^2}$$

which is identical to formula (2.7) obtained through classical physical derivation, so we have verified it experimentally, “empirically”.

Exercise 3.

The truth of the 1st law can be verified easily visually by running the simulation, since elliptical trajectories appear. *A handy feature of DS is that the coordinates of the current position of the crosshair moved with the mouse in the graphics window can be seen in the status bar at the bottom of* *the screen.* One of the focal points of the ellipse is the origin, as we put the centre of the force field there.

How could we find the other focal point of the ellipse?

Once the two focal points are found, we can use the definition of the ellipse to verify the elliptic character, according to which the ellipse is the locus of points in plane whose sum of distances from the two focal points is constant.

We got the possibility of verifying the 2nd law at the end of the interpretation of the vector product described above, according to which it is the size of the area swept out per unit time, $\dot{T} = \frac{1}{2}\left|u \cdot y - v \cdot x\right|$, whose value is calculated and printed out in the text data window in each time step. During the run, we can see that the written value is constant in time, so we have verified the law "experimentally".

In order to verify the 3rd law, run the simulation for a force field with a set *GM* parameter with different *x_ini*, *D* and *v_ini* initial conditions, that is, draw elliptical trajectories in the given force field (the orbits of different "planets" for a given "Sun"). For each setting, try to determine the length of the semimajor axis *a* of the resulting ellipse and the orbital period *T* empirically.

To estimate the value of *a*, use the crosshair coordinate display by reading the coordinates of the two furthest points of the ellipse, calculate their distance using Pythagoras' theorem, and the value of *a* is half of that distance.

To estimate the orbital period *T*, use the *First value* and *Last value* settings in the *Range…* submenu dialog of the *Edit* menu, e.g. set an arbitrary value for *First value*, then change the value of *Last value* until the drawn trajectory closes (Fig. 12), then the difference between the first value and the last value is the orbital period.

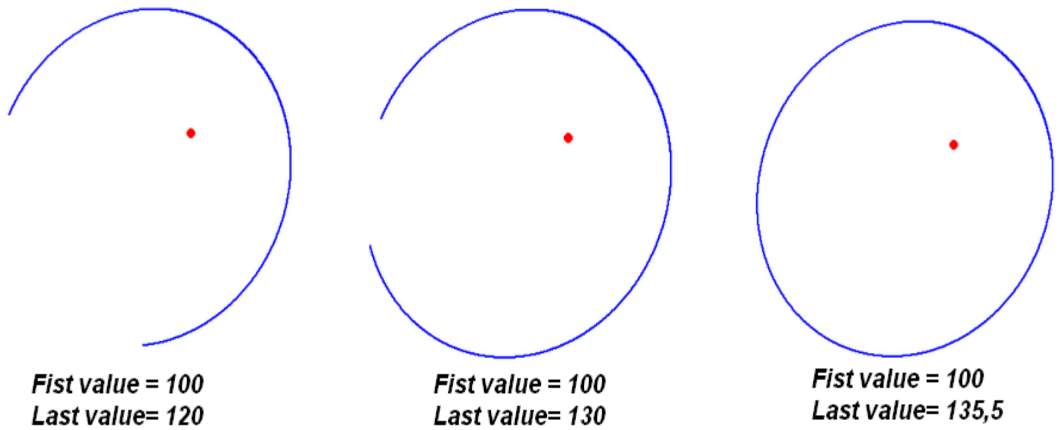

Figure 12. Determining the orbital period *T* "experimentally" in the simulation (*T* = 35.5)

Based on the "measured" *a* and *T* values it is clearly seen that $\frac{a^3}{T^2} = constant$, more precisely that

$$\frac{a^3}{T^2} = \frac{G \cdot M}{(2\pi)^2} = \frac{GM}{(2\pi)^2}$$ is true.

Thus, we have verified Kepler's laws "experimentally", empirically.

Exercise 4.

The typical graphical display of the simulations obtained by running the *Rutherford_sim.ds* problem file is shown in the figure below.

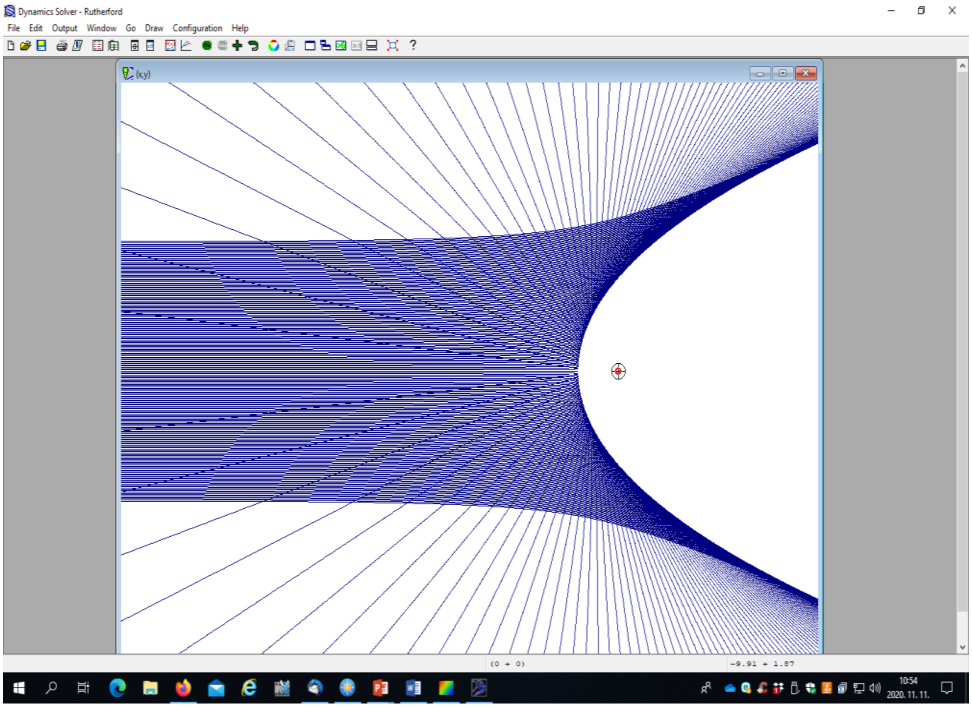

Figure 13. Scattering pattern obtained through DS simulation

The scattering pattern obtained corresponds to the pictures given in literature. By changing the parameters, the angular distribution changes visibly, that is, the properties of the scattering centre can be deduced from the exact image, which is the essential (Nobel Prize winning) result of Rutherford's measurement.

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