Investigation the chaotic properties of the Duffing-oscillator with Dynamics Solver

The electronic materials can be downloaded from [4].
If you wish to study any real object, you must notice that it will have such a great number of properties, which is way beyond our capacity to take into account and describe to the full extent. This is why real objects are 'stripped naked', in other words only characteristics important for the purposes of the study are retained and only the laws of mathematics describing the relationship between the quantities specifying these so called relevant properties are identified. The set or correlations obtained this way, when taken together, are called the mathematical model of the object in question. The goal is usually to look for the time dependence of the relevant state variables of the system. A special significance is given in such a description to the *rate of change of certain state variables over time*.

NOTE! If you are bothered by the too complicated mathematical notions and signs, you can safely use the reading throughout the discussions which follow here: wherever you can see in a formula something like $\dot{y}$, or $\frac{dy}{dt}$, read as "rate of change of *y* over time" (for instance, formula (2) can be read as "the rate of change of the speed *v* over time is *a* which stands for acceleration"). You will see that *all equations of motion providing the development of a system over time can be brought to a form where "the rate of change of a quantity over time" stands on the left hand side of the equation and a mathematical expression specifying this rate of change can be found on the right hand side of the formula.* It is very important to highlight that it is sufficient to enter only the mathematical expressions on the right hand side to any numeric computer programme calculating the changes over time of any system, that is the reading suggested here is not only sufficiently precise but also the most practical way possible!

Let's try to provide an exact mathematical definition to the rate of change over time! Let's see the quantity *y*, with a time dependence specified by the function *y(t)*. Mark the changes in the value of *y* during a given short period of time Δt by Δy. The rate of change – very apparently – will be specified by the "change velocity" $\frac{\Delta y}{\Delta t}$, which is called in maths as the differential quotient. It is also clear that the momentary value of the change will be provided by the differential quotient the more precisely, the shorter Δt period of time is taken into consideration. Extremely short (so-called infinitesimal) periods of time in maths are marked as d*t*, the change of *y* during this time is d*y*. Using these variables, *the momentary rate of change* $\frac{dy}{dt}$*, also called differential quotient and is also indicated as* $\dot{y} = \frac{dy}{dt}$.

The process of model creation is presented below through the example of a real mechanic system which can be discussed in possession of relatively simple, basic level knowledge of physics: the motion of a body oscillating along a straight line with a mass of *m* will be studied. The outline of the mechanic system can be found on Figure 1 below.

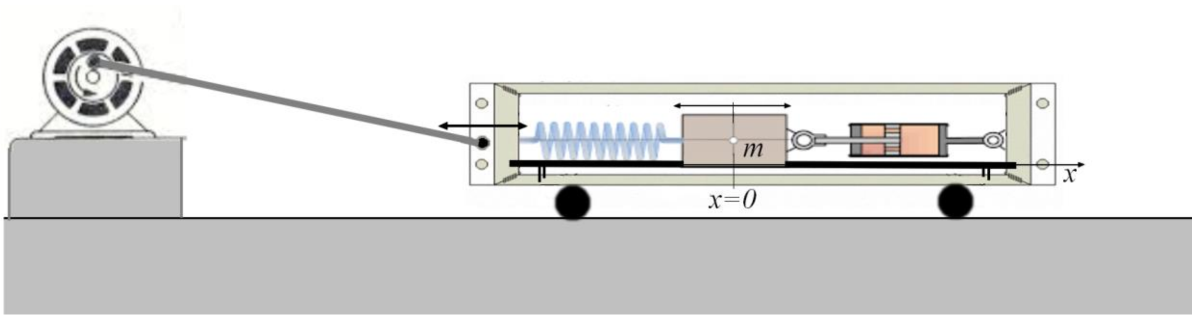

Figure 1: the schematic representation of the mechanic system tested

The car rolling on a horizontally laid pair of rails is jerked using an eccentric disk. A pair of horizontally laid rails is fastened on the car itself, with a body having a mass of *m* on it, moving easily on the rails. The left side end of the body is fixed to the front wall of the car using an untensioned spring, and the right side end of the body is fixed to the rear wall using a piston moving in a hydraulic cylinder. The line of the rail is considered hereinafter as the *x* axis of the coordinate system and the location of the centre of mass of the body under untensioned conditions on the rails will be denoted as $x = 0$ point, in other words the origin of coordinates (and, with it the coordinate system itself) will be fixed to the car.

It is clear that the centre of mass of the body will carry out a uniform rectilinear motion in the proximity of the $x = 0$ point, and this is the motion we intend to describe. One of the relevant state variable in the system is obviously the displacement of the body relative to *x* (the origin of the coordinates), the momentary value of which is specified using the function *x(t)*. The precise mathematical specification of the change over time in the displacement *x(t)* shall be the *velocity* defined by the differential quotient:

$$\text{'rate of change of } x \text{ over time'} = \frac{dx}{dt} = \dot{x} = v \,. \qquad (1)$$

Speed, hence, is in fact nothing else but the ‘rate of change’ defined as the quotient of *dx* displacement during *dt* time and *dt* time.

Apparently, the speed marked *v(t)* of the body in question may also change over time. Stepping one more step further mentally, *acceleration*, which provides the rate of change of the *v(t)* function of speed can be defined the same way as it happened in the case (1):

$$\text{'rate of change of } v \text{ over time'} = \frac{dv}{dt} = \dot{v} = a \,. \qquad (2)$$

Let’s try to think it over what kind of information will you need to calculate the change over time in a mechanic system! Take a body with the mass *m* moving along a straight line as an example. If you know the *x(t)* coordinate and *v(t)* velocity of the body at the moment *t*, its subsequent coordinate after the lapse of an (“appropriately”) small Δt time can be calculated in the form $x(t+\Delta t) = x(t) + v(t)\cdot\Delta t$ (the calculation will be the more accurate the less Δt is). Well, yes, but what will be the velocity at this point? You can obtain this value with the formula $v(t+\Delta t) = v(t) + a(t)\cdot\Delta t$ which is analogue to the former one, provided the acceleration *a(t)* of the body at the moment *t* is known. *Acceleration a of a body, then, will be proportional to the resultant force of the forces it is exposed to, as the Second Law of Newton postulated*:

$$m \cdot \vec{a} = \sum \vec{F} \,,$$

which, using he definition (2) will be (in one dimension):

$$\dot{v} = a = \frac{1}{m} \sum F \,, \tag{3}$$

It turns out from the line of thought referred to above that you will need to know the *x0 = x(t)* initial position and the *v0 = v(t)* initial velocity in order to be able to start the calculation.

The general dynamic model, the so-called *Duffing-oscillator* of the body carrying out oscillating motion in the system depicted on Figure 1 will be built up below in four steps, so that the dynamic effects featured on the right hand side of the equation (3) are expanded on a step by step basis:
(1) linear spring force,
(2) taking into account the friction (viscous) force impeding the motion,
(3) expanding non linear spring force,
(4) adding a periodical external exciting force.

<u>(1) Linear spring force</u>
The size of the *spring force* shall be proportional with the momentary displacement of the body (in fact, with the stretching of the spring), and its direction is opposite to the displacement (it both ‘pulls’ and ‘pushes’), in other words:

$$F_r = -D \cdot x \,, \tag{4}$$

where *D* shall be the *so-called spring constant* (its value is the higher the stronger the spring is). Putting the force (4) on the right hand side of the formula (3) you will get:

$$a = -\frac{D}{m} \cdot x \,, \tag{5}$$

and if you put the definition (1) providing the rate of change of the displacement *x* beside it, you will get the equations of motion describing the changes over time:

$$\begin{cases} \text{'rate of change of } x \text{ over time'} = v \\ \text{'rate of change of } v \text{ over time'} = -\dfrac{D}{m} \cdot x \end{cases} \tag{5a}$$

(From the point of view of mathematics, (5a) is a set of differential equations, the solution of which is provided by the *x(t)* and *v(t)* functions which comply with the equations of (5a). The set of (5) is a very simple type of differential equations, which can be solved ‘manually’ just as well, but this is not the purpose of this current discussion.) In this attempt, as it was done before using the Excel spreadsheets, *the x(t) and v(t) functions will be identified and studies with the help of a computer and the numeric method.* Numeric mathematical methods will be covered in a separate chapter, it was assumed in this case that the computer programme necessary for this such as the free *Dynamics Solver* programme (http://tp.lc.ehu.es/jma/ds/ds.html) is available, which is also described in our DS_brief_tutorial.pdf. The following functions need to be applied for the most fundamental level of use of an already completed **.ds problem-file*: run with the icon, stop with the icon, continue with the icon, delete graphic windows using the icon. The display of the parameter table with the icon, and the display of the initial conditions table will take place with the use of the icon. At this point we recommend to try the *harmonic_oscillator.ds* programme!

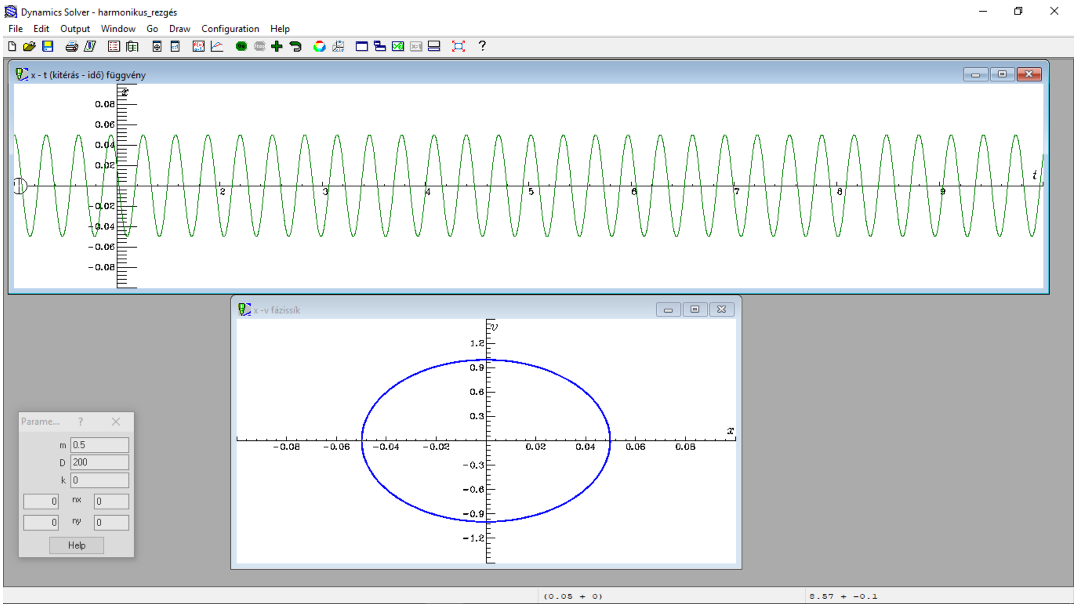

Figure 2: the image of the linear harmonic oscillating motion in the programme Dynamics Solver (m = 0.5, D = 200)

Let's assume that for instance $m = 0.5$ and $D = 200$, with the initial conditions of $x0 = 0.05$ and $v0 = 0$, respectively (in SI units of measurement, use decimal points in the programme!), and a slight change to the values of the *m* mass and the *D* spring constant parameters is recommended. (The parameter *k* is also included. This will be explained in the next section, for the time being, let it be a zero value). You can view the results of the run of the opened problem file in two graphic windows. In the upper window the *x(t)* displacement function is plotted. A (sinusoidal) periodic motion, the well-known *harmonic oscillatory motion* is clearly discernible. The motion of the so-called *phase-point*, provided by the values taken by the *x* displacement and *v* velocity at any moment are shown in the lower window in the so-called *x-v phase plane*, i.e. the curve drawn, along which the phase point moves on. It is called a *trajectory*. This time, the trajectory is a closed curve (i.e. a circle or ellipse), which is also a way to illustrate periodical motion, since the phase point goer round and round the curve, i.e. a so-called *limit cycle* is formed.

<u>(2) Taking friction into account</u>

In the next step, the friction force, impeding the motion in question will be included in the description, which is, in fact, the *viscous friction force* impacting the piston moving in the hydraulic cylinder, since it is proportional with the momentary *v* velocity of the body (think of a car: the higher its speed of travel, the larger is the braking force exerted on it by the air), in other words:

$$F_k = -k \cdot v \,, \tag{6}$$

where $k > 0$ *is the viscous attenuation coefficient*.

Putting the forces (4) and (6) in the right hand side of the (3) Newton-law:

$$a = -\frac{D}{m} \cdot x - \frac{k}{m} \cdot v \,,$$

and adding the definition (1) providing the rate of change of the displacement *x* beside it, you will get the dynamic set of equations describing the changes over time:

$$\begin{cases} \text{'rate of change of } x \text{ over time'} = v \\ \text{'rate of change of } v \text{ over time'} = -\dfrac{D}{m} \cdot x - \dfrac{k}{m} \cdot v \end{cases} \tag{7}$$

Let's run the [harmonic_oscillator.ds] programme again, but this time provide the *k* friction parameter a value other than zero, let it be for instance $k = 0.5$.

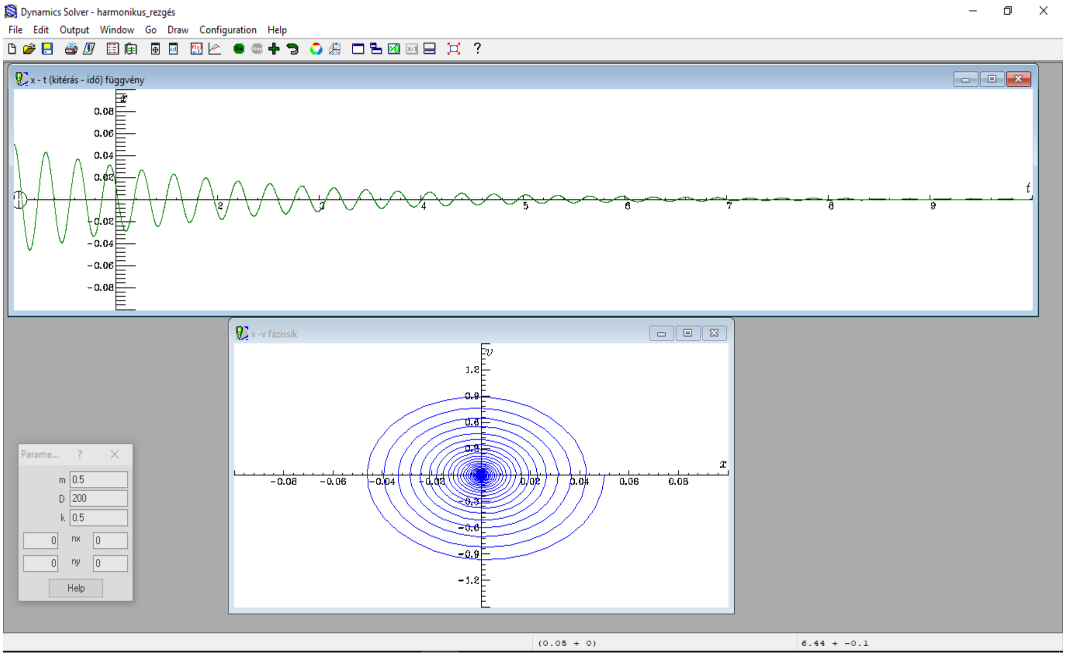

Figure 3: the image of an attenuating oscillatory motion in Dynamics Solver (m = 0.5, D = 200, k = 0.5)

The figure obtained speak for themselves again: an attenuating oscillatory motion appears. The displacement-time function is obviously a sinusoidal oscillation with a well perceivable exponentially declining amplitude, while the trajectory on the displacement-velocity phase plane it such a spiral which leads into a *stable state called fixed point* (in this case, the origin of the coordinates, that is in a state with zero displacement and velocity).

<u>(3) Expansion of the non-linear spring force</u>

Taking the material properties of the spring into account just as well, you can typically distinguish springs which *harden* and springs which *soften* with stretching. Mathematically these impacts can be included in the force dependence by using a new member proportional with x^3 (an x^2 member makes no sense in physics, since it is independent from the sign of x, although it is expected from an elastic force that it always be opposite to the displacement), therefore, instead of (4), now:

$$F_r = -D \cdot x - E \cdot x^3 , \qquad (8)$$

where *E is the so-called anharmonic factor which is* $E > 0$ *for hardening and* $E < 0$ *for softening springs*.

Putting the elastic force (8) and the friction force (6) in the right hand side of the (3) Newton-law:

$$a = -\frac{D}{m} \cdot x - \frac{E}{m} \cdot x^3 - \frac{k}{m} \cdot v ,$$

and adding the definition (1) providing the rate of change of the displacement x beside it, you will get the dynamic set of equations describing the changes over time:

$$\begin{cases} \text{'rate of change of } x \text{ over time'} = v \\ \text{'rate of change of } v \text{ over time'} = -\dfrac{D}{m} \cdot x - \dfrac{E}{m} \cdot x^3 - \dfrac{k}{m} \cdot v \end{cases} \qquad (9)$$

Run the [anharmonic_oscillator.ds]() programme this time, where the new E parameter will also appear, and choose for instance a value of $E = 8*10^6$. Without friction ($k = 0$) still a periodical motion (limit cycle) appears, while in the case of a friction force (pl. $k = 1$) the oscillatory motion's frequency changes over time with an attenuating amplitude (Figure 4).

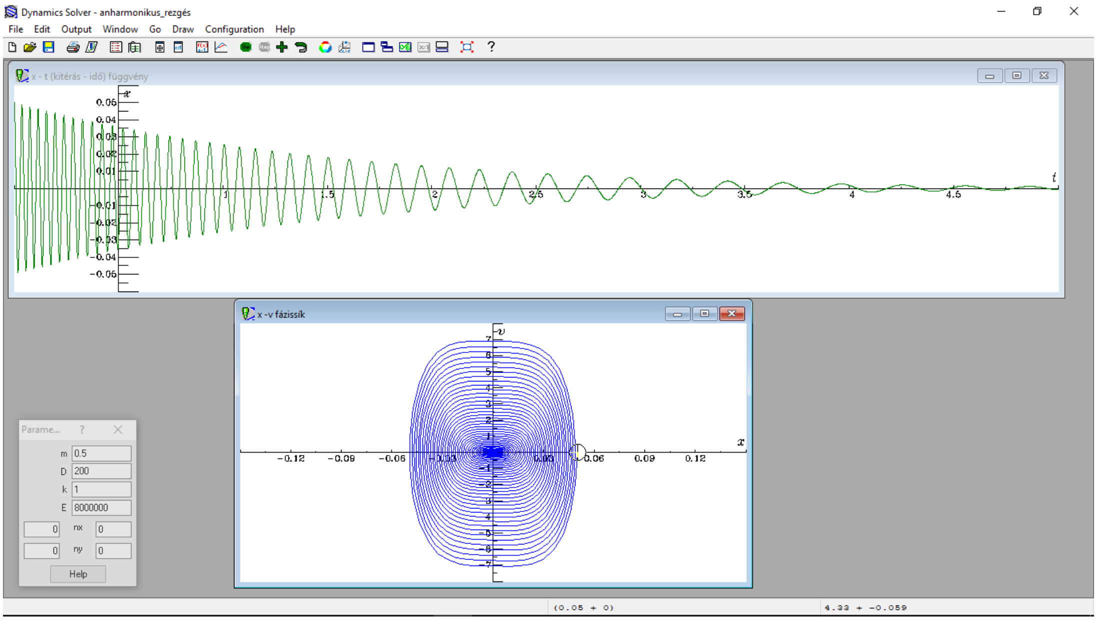

Figure 4: the image of an anharmonic attenuating oscillatory motion in Dynamics Solver

(4) Addition of an external periodical exciting force

The environment of an oscillating body may exert a force impact on the body, and in many cases such an impact is of periodical nature. (In the implementation method seen on Figure 1 the external excitation is provided by the eccentric disk.) The impact of the eccentric disk can be taken into account with the use of the *harmonic function with F_0 force-amplitude and T cycle time*:

$$F_g = F_0 \cdot \cos\left(\frac{2\pi}{T}t\right). \qquad (10)$$

Putting the elastic force (8), the friction force (6) and the exciting force (10) in the right hand side of the (3) Newton-law:

$$a = -\frac{D}{m}\cdot x - \frac{E}{m}\cdot x^3 - \frac{k}{m}\cdot v + \frac{F_0}{m}\cdot\cos\left(\frac{2\pi}{T}t\right),$$

and adding the definition (1) providing the rate of change of the displacement x beside it, you will get the dynamic set of equations describing the changes over time:

$$\begin{cases} \text{'rate of change of } x \text{ over time'} = v \\ \text{'rate of change of } v \text{ over time'} = -\dfrac{D}{m}\cdot x - \dfrac{E}{m}\cdot x^3 - \dfrac{k}{m}\cdot v + \dfrac{F_0}{m}\cdot\cos\left(\dfrac{2\pi}{T}t\right) \end{cases} \qquad (11)$$

At this point, however, you must think it over, how many variables you need to specify the complete state of the system! From a physical perspective it is not difficult to understand that the provision of the momentary *x(t)* displacement and *v(t)* velocity is not enough any more in the case of an external application of a force if you want to determine the value of the *x(t+dt)* displacement (as well as the *v(t+dt)* velocity) which can be expected in the next moment, since it can obviously be influenced by the dynamic effect of the external force acting at the moment *t*, which is characterised by the following phase in the case of the periodical excitation (10):

$$\varphi = \frac{2\pi}{T}t. \qquad (12)$$

This is then, how the *phase space of the excited oscillation will be expanded to three dimensions: beside x displacement and v velocity the φ excitation phase will be included among the state variables*. According to (12) φ phase is a linear function of *t* time, consequently the steepness of the 'rate of change' (i.e. the differential quotient) line is thus:

$$\text{'rate of change of } \varphi \text{ over time'} = \frac{2\pi}{T}. \tag{13}$$

When (11) is completed with the equation (13) *the set of differential equations providing the dynamics of the periodically excited frictional anharmonic oscillation, in other words of the Duffing-oscillator can be obtained*:

$$\begin{cases} \text{'rate of change of } x \text{ over time'} = v \\ \text{'rate of change of } v \text{ over time'} = -\dfrac{D}{m} \cdot x - \dfrac{E}{m} \cdot x^3 - \dfrac{k}{m} \cdot v + \dfrac{F_0}{m} \cdot \cos\varphi \\ \text{'rate of change of } \varphi \text{ over time'} = \dfrac{2\pi}{T} \end{cases} . \tag{14}$$

Figure 5 shows an actual experimental setup implementing the Duffing-oscillator (external excitation is provided here by a vibrating (shaking) table instead of the car jerked with the eccentric disk), and additional experimental arrangements of the model can be seen on the video Duffing_magnets.mov.

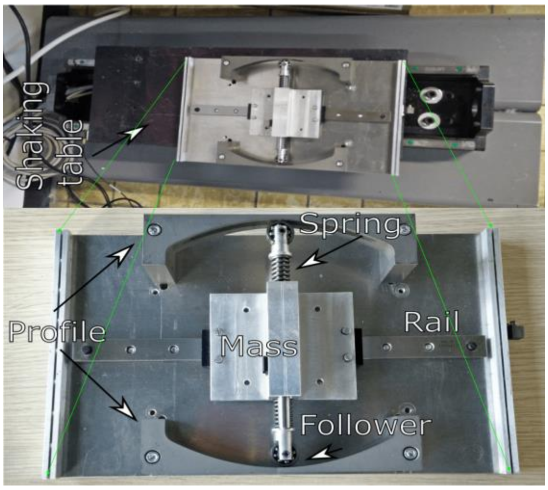

Figure 5: the picture of a Duffing-oscillator [1] implemented in an experimental arrangement

With the help of the formula (14) the mathematical model wanted is now obtained, which is perfectly suitable in this form to provide the input to some computer programme: in fact, numeric algorithms need the expressions defining the rate of change of the variables over time (found on the right hand side of these equations) and, of course, the values of the parameters included in them.

You can use, for instance, the Dynamics Solver programme to carry out the numeric simulation. Let's run the *Duffing-oscillator.ds* problem file on it (the preparation of the problem file is introduced from step to step in a separate chapter in details). The respective values of the selected physical parameters (specified in SI units of measurement) may be as follows (for the order of magnitude of the range of parameters see for instance in [2] and [3]):

$$m = 0{,}5\ kg\ ,\ D = 2 \cdot 10^2\ \frac{kg}{s^2}\ ,\ E = 8 \cdot 10^6\ \frac{kg}{m^2 s^2}\ ,\ k = 1\frac{kg}{s}\ ,\ F_0 = 50\frac{kg \cdot m}{s^2}\ \text{ és } T = 0{,}3\ s.$$

You only enter the numeric values in the programme, under the agreed assumption that all values are to be understood in SI units of measurement. From this it will also follow that the respective variables (time, displacement, velocity and exciting phase) will also appear in SI units! A copy of the run screen can be seen on Figure 6.

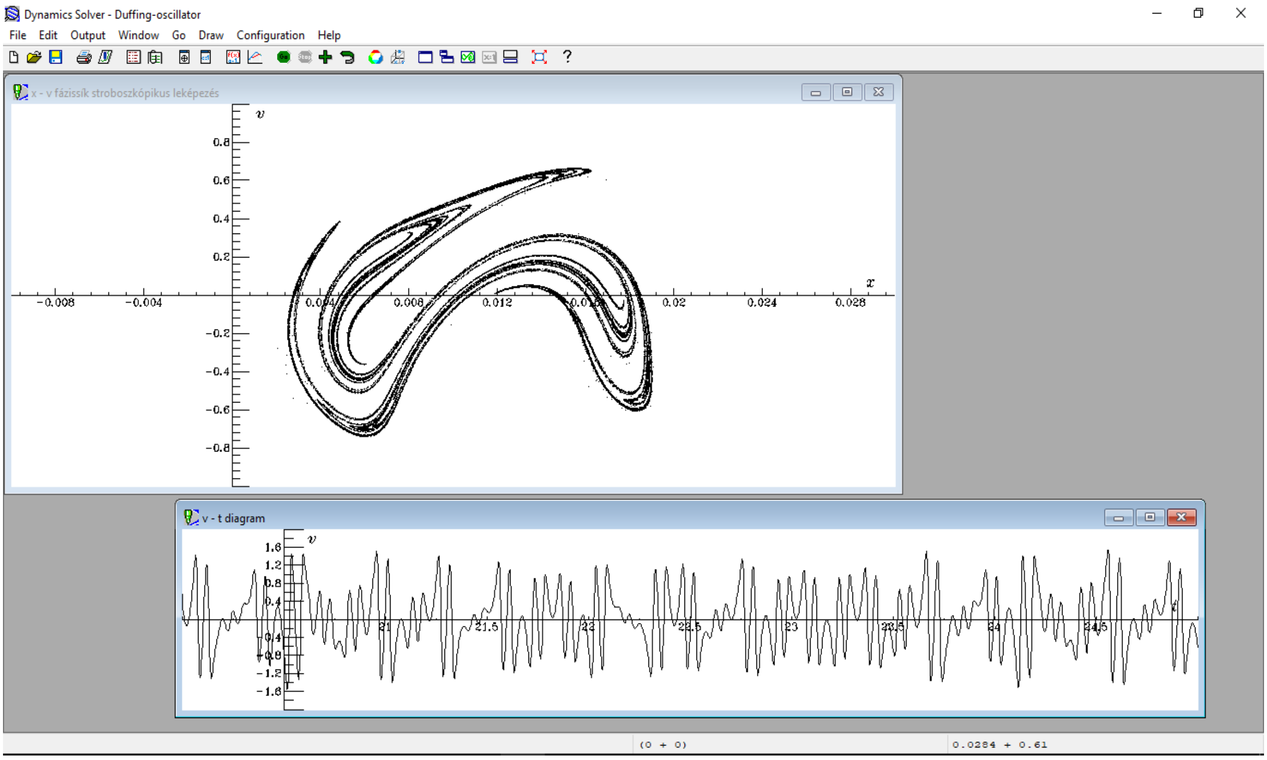

Figure 6: chaotic behaviour of the Duffing-oscillator in Dynamics Solver

The lower graphic window displays the *v(t)* velocity-time function and it can be clearly seen that it does not present any regular behaviour, the *chaotic nature appears*. In the upper graphic window the *x-v* (displacement-velocity) phase plane is displayed again, but using a specific representation this time, the so-called *stroboscopic imaging* because of the periodical excitation, which is in its essence discrete time sampling in the identical phase time moments of the exciting force. (As if a series of photographs would be taken using the stroboscope lamp's light signals flashing with a *T* cycle time. To better understand how it works, see the video stroboscop.mp4). The strange shape plotted in the *x-v* phase plane using the stroboscopic imaging technique will be the *chaotic attractor of the Duffing-oscillator*, showing an interesting geometric structure, bearing the characteristics of a *fractal pattern* (the exciting world of fractals is shown in a separate paper).

With the help of the Dynamics Solver an additional key feature of chaos can be studied as well in a simple way: this is the *extreme sensitivity* of the behaviour of a chaotic system over time *reflecting to the initial conditions*. This means that very different time dependence can be experienced when the system is set off from very close initial conditions with the same values of the parameters, in other words the customary approach 'little initial error resulting in a little deviation of the result' will not be met in this case. The most spectacular way of illustration of this extreme level of sensitivity is the plotting of the so-called *ensemble diagram*. This diagram depicts the changes over time of a variable in the chaotic system starting from multiple initial conditions very close to each other. Let's run the *Duffing-oscillator_ensemble.ds* problem file! When the parameter values defined above are entered into the programme, it shows the *v(t)* velocity function in 5 different, but very close initial x_0 displacement values (in order with the actual values of -0.0006; -0.0003; 0; +0.0003; +0.0006), while the initial value of the other two variables of the system is always identical ($v_0 = 0$ and $\varphi_0 = 0$).
The screen image received can be viewed on Figure 7. It is clearly seen that the curves providing the changes of velocity over time run in parallel for a while, then diverge entirely, in other words the minimum differences in the initial values result in entirely different behaviour after the lapse of a characteristic time period.

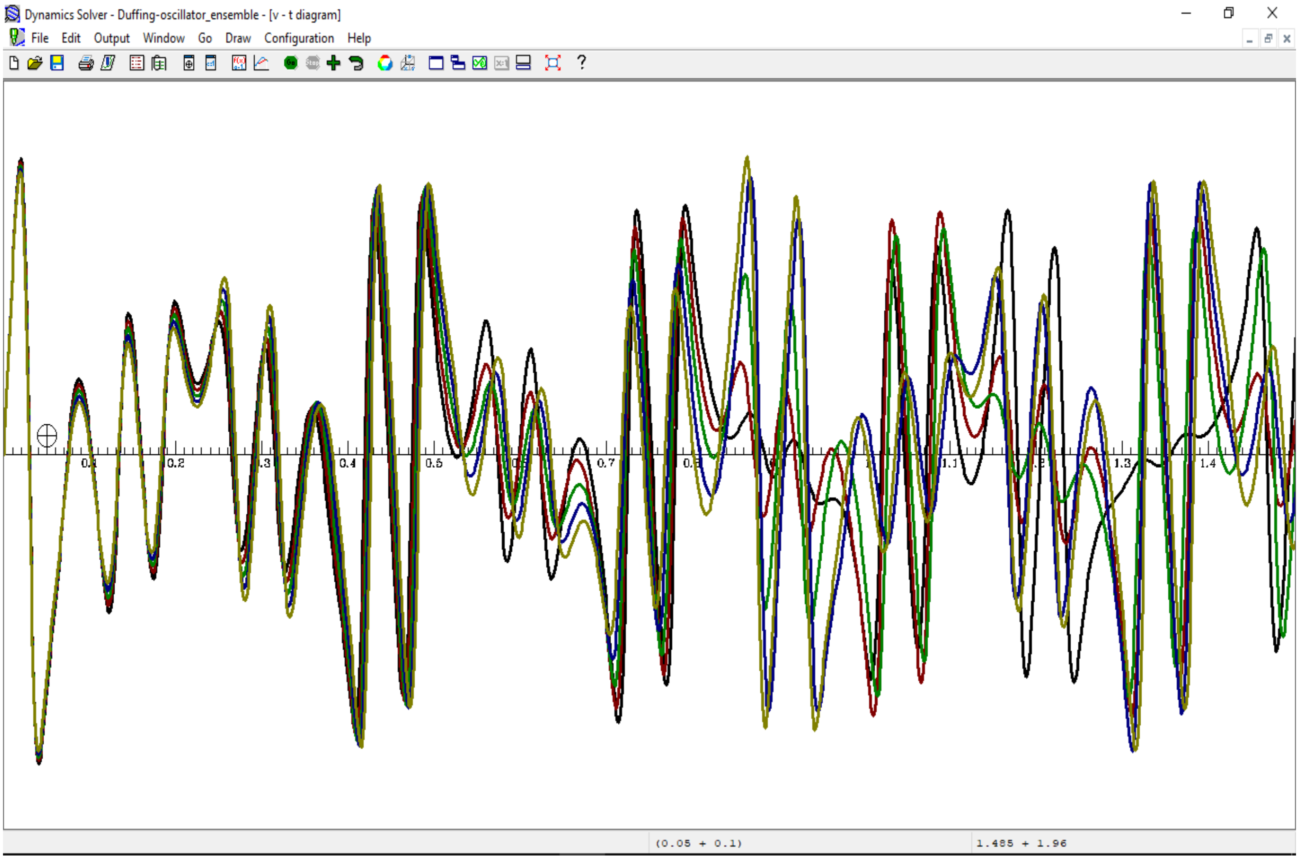

Figure 7: ensemble diagram of the Duffing-oscillator in Dynamics Solver

Based on these results the characteristic features of the ***chaos*** phenomenon can be postulated precisely:

- ***change over time (i.e. motion) of a system which can be described with simple (low degree of freedom) and unanimous (deterministic) regularities (that is, using a couple of non-linear dynamic equations),***
- ***irregular (non-periodical, complicated) behaviour,***
- ***long term behaviour in the phase space fully specifying the change over time is characterised by a kind of geometric structure and order (the attractors in the phase space show fractal geometry),***
- ***due to the excessive growth of the error in the initial state (in other words, extreme sensitivity to the initial conditions) predictions for a long term are practically impossible to make (only a probability description can be given).***

Finally a line of thought is presented here for the parametering of the exciting force. Provided external excitation takes place in a way shown on Figure 1, the eccentric disk would jerk the car with the harmonic oscillatory motion described by the $x_K(t) = A_0 \cdot \cos\left(\frac{2\pi}{T}t\right)$ displacement-function, A_0 amplitude and T cycle time. In case of a harmonic oscillation acceleration is proportional (and opposite direction) with displacement, therefore the time function of the acceleration of the car will be as follows:

$$a_K(t) = -\left(\frac{2\pi}{T}\right)^2 \cdot x_K(t) = -\left(\frac{2\pi}{T}\right)^2 \cdot A_0 \cdot \cos\left(\frac{2\pi}{T}t\right).$$

Since the body with m mass is found in the accelerating coordinate system fixed to the car, it is exposed to the inertial force $F_t = -m \cdot a_K$, assuming the role of the exciting force:

$$F_g(t) = -m \cdot a_K(t) = m \cdot \left(\frac{2\pi}{T}\right)^2 \cdot A_0 \cdot \cos\left(\frac{2\pi}{T} t\right).$$

This formula compared with the formula of the exciting force (10) it can be concluded that in the case exciting takes place with an eccentric disk, the following applies:

$$F_0 = m \cdot \left(\frac{2\pi}{T}\right)^2 \cdot A_0,$$

of which, the A_0 amplitude of jerking:

$$A_0 = \frac{F_0 \cdot T^2}{m \cdot (2\pi)^2}.$$

Substituting the parameter values specified above, the amplitude applicable to the jerking provided by the eccentric disk will be A_0 = 0.228 m.

Appendix: Elimination of dimensions

Below, a detailed explanation follows on the method of the so-called *elimination of dimensions* on the example of the Duffing-oscillator, the detailed mathematical derivation can be skipped.

With the equation (14) in fact the wanted mathematical model has already been obtained, but an important aspect was neglected up to this point. Now it is time to reconsider: the role of units of measurement. Physical quantities do have a numeric value but they must have a unit of measurement just as well, for instance in the motion equations of the Duffing-oscillator (14) under consideration there are 1+3 variables (t time as an independent variable and x displacement, v velocity and φ phase as dependent variables) and 6 parameters (m, D, E, k, F_0 and T), which have SI units as follows:

$$[t] = s \text{ és } [x] = m,\ [v] = m/s,\ [\varphi] = 1 \text{ for variables,} \qquad \text{(F1..a)}$$

and $$[m] = kg,\ [D] = \frac{kg}{s^2},\ [E] = \frac{kg}{m^2 s^2},\ [k] = \frac{kg}{s},\ [F_0] = \frac{kg \cdot m}{s^2} \text{ és } [T] = s$$

for parameters, (F.1.b)

However, a computer works with numbers only, units of measurement are uninterpretable for it. T solve the problem, the customary procedure is elimination of the dimensions, in other words the introduction of new variables without a unit of measurement.

As a first step, basic values are chosen for the two basic variables: L base length for displacement and T base time for time, by which the new dimensionless variables x' és t' are introduced:

$$x = L \cdot x' \text{ és } t = T \cdot t',$$

Using these, the differential changes will be as follows:

$$dx = L \cdot dx' \text{ és } dt = T \cdot dt',$$

therefore:

$$\dot{x} = \frac{dx}{dt} = \frac{L \cdot dx'}{T \cdot dt'} = \frac{L}{T}\frac{dx'}{dt'} = \frac{L}{T}\dot{x}' \text{ and } \dot{v} = \frac{dv}{dt} = \frac{d\left(L/T \cdot v'\right)}{d\left(T \cdot t'\right)} = \frac{L}{T^2}\frac{dv'}{dt'} = \frac{L}{T^2}\dot{v}'.$$

Substituting them in the middle equation of (14):

$$\frac{L}{T^2}\dot{v}' = -\frac{D}{m}\cdot L\cdot x' - \frac{E}{m}\cdot L^3\cdot x'^3 - \frac{k}{m}\cdot\frac{L}{T}v' + \frac{F_0}{m}\cos\left(\frac{2\pi}{T}T\cdot t'\right),$$

multiplying both sides of the equation with the expression $\frac{T^2}{L}$:

$$\dot{v}' = -\frac{D}{m}\cdot T^2\cdot x' - \frac{E}{m}\cdot T^2L^2\cdot x'^3 - \frac{k}{m}\cdot T\cdot v' + \frac{F_0T^2}{mL}\cos\left(2\pi\cdot t'\right),$$

It is worth to consider, how to select the arbitrary base values for *T* and *L*. Selection of *T* is almost trivial (even the letter sign is identical), let's choose *T* cycle time of excitation. As to the selection of *L*, a possibility readily at hand will be to choose it so that the two first members on the left hand side of the equation (that is, those members which are dependent on the displacement) would have an identical coefficient, that is:

$$\frac{D}{m}\cdot T^2 = \frac{E}{m}\cdot T^2L^2,$$

from which:

$$L = \sqrt{\frac{D}{E}}.$$

Having made these choices, (14) will now be transformed into:

$$\begin{cases} \dot{x}' = v' \\ \dot{v}' = -\frac{D}{m}\cdot T^2\cdot\left(x' + x'^3\right) - \frac{k}{m}\cdot T\cdot v' + \frac{F_0T^2}{m}\sqrt{\frac{E}{D}}\cos\left(2\pi\cdot t'\right) \\ \dot{\varphi}' = \frac{2\pi}{T} \end{cases}.$$

Finally, for reasons of convenience (i.e. laziness) denotation with a comma will be omitted for the new variables, and thus the end result will be as follows:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -A\cdot\left(x + x^3\right) - B\cdot v + C\cdot\cos\left(\varphi\right) \\ \dot{\varphi} = 2\pi \end{cases}. \qquad \text{(F.2.)}$$

where:

$$A = \frac{D}{m}T^2\ ,\ B = \frac{k}{m}T\ \text{és}\ C = \frac{F_0T^2}{m}\sqrt{\frac{E}{D}}\ , \qquad \text{(F.2..a)}$$

which *parameters are also lacking any unit of measurement* (!), as you can easily make sure on the basis of (F.1.b). An additional observation of fundamental importance is that *during the elimination procedure of dimensions the number of model parameters dropped from* 6 to 3, in other words all systems where the *A*, *B* and *C* parameters are identical will be *equivalent in terms of mathematical behaviour (mathematically isomorphic),* irrespective of the actual values given to the original *m*, *D*, *E*, *k*, F_0 and *T* parameters!

The (F.2) set of differential equations is a standard form which can be specified as an input to any arbitrary numeric computer programme (such as the Dynamics Solver) (in fact, only the controlling function on the right hand side of (F.2) need to be entered), and thus suitable for a dynamic testing procedure of the model (including, for instance, the examination of the chaos phenomenon). Let's run the *Duffing_dimless.ds* problem file prepared according to (F.2)! Selected physical parameter values should be the same as defined above:

$$m = 0,5\ kg\ ,\ D = 2\cdot 10^2\,\frac{kg}{s^2}\ ,\ E = 8\cdot 10^6\,\frac{kg}{m^2 s^2}\ ,\ k = 1\frac{kg}{s}\ ,\ F_0 = 50\frac{kg\cdot m}{s^2}\ \text{és}\ T = 0,3\ s\,,$$

by which, according to (F.2.a), the parameters lacking any dimensions will be:

$$A = 36\,,\ B = 0,6\ \text{és}\ C = 1800\,.$$

A copy of the run screen can be seen on Figure 8.

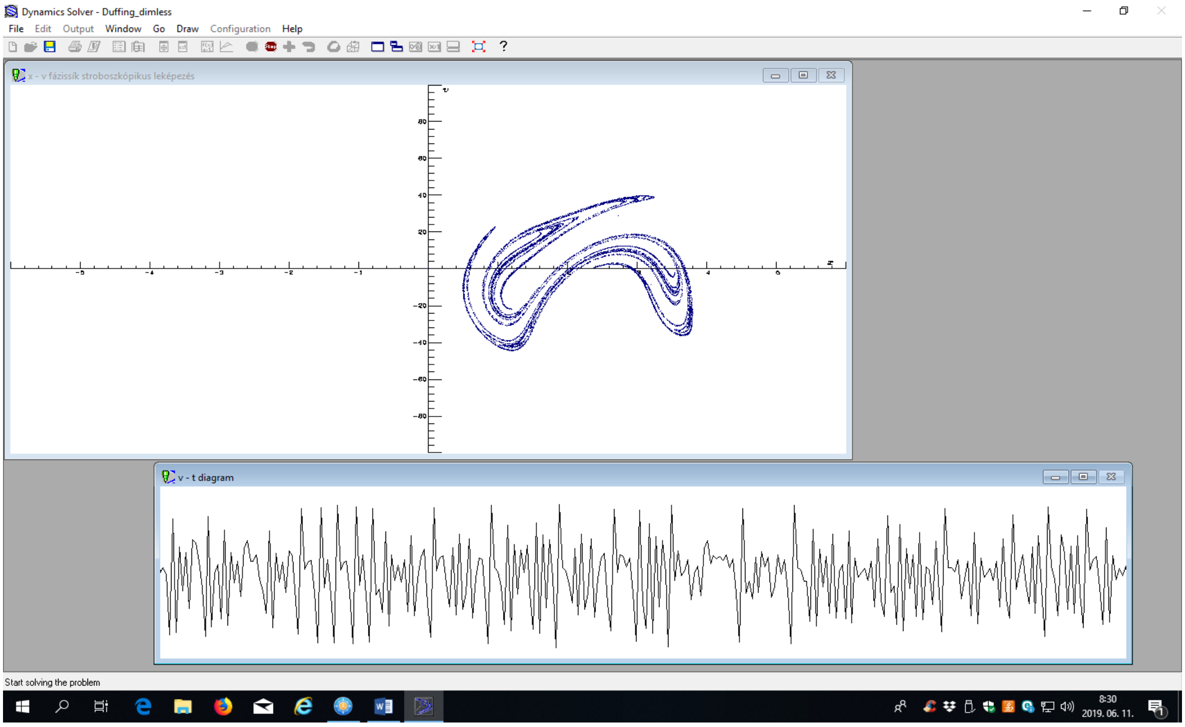

Figure 8: chaotic behaviour of the Duffing-oscillator in the Dynamics Solver

References:

[1] https://arxiv.org/pdf/1805.03499.pdf

[2] https://www.researchgate.net/publication/266674956_Experimental_investigation_of_vibration_attenuation_using_nonlinear_tuned_mass_damper_and_pendulum_tuned_mass_damper_in_parallel

[3] https://www.researchgate.net/publication/224912137_Rheology_of_fluids_measured_by_correlation_force_spectroscopy?_sg=cu0yZb9ga62kxRUhZN2omQ_Ssfi_NWiFZKYnR2xkJVHoA1RF7BMXkvEaB7Yct380Ix0cAZuipA

[4] http://fiztan.phd.elte.hu/letolt/Duffing_DS.zip