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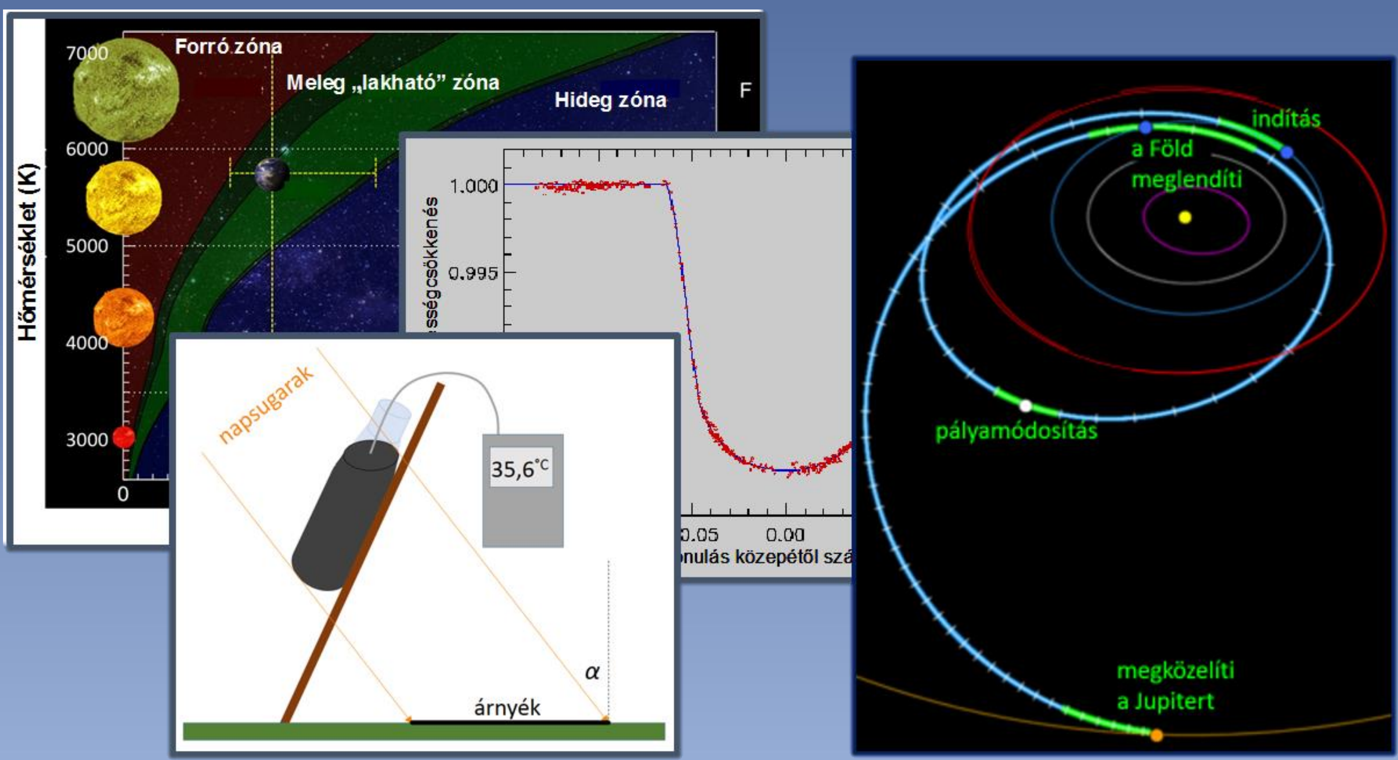
EXOPLANETS AND SPACECRAFT

Exercises in astronomy and space exploration for high school students



ELTE DOCTORAL SCHOOL OF PHYSICS

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Exercises in astronomy and space exploration for high school students

ELTE DOCTORAL SCHOOL OF PHYSICS BUDAPEST

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Introduction

It is important that scientific results are mentioned in lessons, in many cases they are more interesting than the creations of human imagination. Without increasing the curriculum this can be achieved best through problems, however, nowadays problems lose ground in the physics lessons. Conceptual physics, that is, physics that avoids using quantitative relationships and tries to achieve the aim of understanding terms and phenomena more deeply is very popular. Its supporters contrast it with the "traditional teaching of physics", which is depicted as a simple statement of formulas followed by substitution into them. (In the physics methodology literature available in English "end-chapter problems" often has negative connotations.)

As one measure of the effectiveness of education is the successful linking of learning with prior knowledge and its application in new situations [1], problem solving is essential for a deeper understanding. Numerous researches prove that experience gained in problem solving has a positive effect on answering qualitative questions successfully as well [2], [3]. The way of learning, the working of science, the difficulties of individual discoveries and their significance in the history of science can be shown only through measurements and calculations.

Therefore, in search for ways in which the modern areas of astronomy can be interpreted in a high school setting, we were searching for challenging tasks that match the colourful, contemporary astronomical researches and achievements but also match the knowledge of high school students, thus facilitate the implementation of problem-based learning. Problem solving is much more than substitution: recognizing the usability of relations and the usability constraints as well as determining what variables are needed and how their values can be obtained from the available information, which are often not given directly, but in tables, graphs or otherwise.

The unconventional tasks listed here may be of interest to students. Arousing interest is also supported by pictures and drawings, but the emphasis is on presenting the real world, so all tasks are based on real measurement data.

The popularity of science-fiction movies and computer games also shows how much the world of distant celestial bodies moves the imagination of young people. For this reason, we expect that the introduction of the exoplanets of distant star systems (for example, a gas giant greater than Jupiter, but orbiting closer to its star than Mercury, so heating up and losing matter like a comet, or a planet similar to planet Tatooine known from Star Wars with two suns shining in the sky) through problems can be suitable for communicating scientific results at high schools. Most problems therefore investigate this topic.

In the discussion of exoplanets, the methods used for their discovery are of utmost importance, so we dedicated a separate chapter to them. Besides the most effective ones, namely the transit and the Doppler methods, we also deal with the astrometry method, where the researchers expect to achieve the appropriate sensitivity through the Gaia space telescope, which is already in operation today. With these methods, the process of scientific cognition can also be traced, in which we first perform observations and collect data with more and more accurate instruments, then compare the results of calculations using models based on these data with the experience.[1]

One of the main directions of the conclusions of planetary properties (such as their surface temperature, atmosphere) is whether it can be imagined that they hold life (similar to life on Earth). In the problems, the issue of habitability is simplified to ensuring that water can exist in liquid state permanently on the exoplanet. The observation the planets of the Solar System from a distance is important for the same reason, suggesting that we are not the only ones who search for extra-terrestrial

[1] As a result of the two decades of research, we now know the essential properties of nearly four thousand exoplanets. From the aspect of school education, it is important to know that the measurement data are public, and the results acquired from them can be browsed in databases that are updated regularly (e.g. exoplanet.eu).

(intelligent) life. The examples illustrate the difficulties in discovering distant planetary systems that arise not only in the case of Earth-type planets, but also in the case of gas giants similar to Jupiter.

Another chapter contains problems related to modern spacecraft. This is not only due to the fact that the exoplanet discovered in Proxima Centauri's habitable zone raises the question of the availability of nearby stars with existing spacecraft. We also want to draw attention to the results of the growing number of space probes and space telescopes in the Solar System. We commemorate one of the oldest space probes, Voyager-1, which has been communicating for 40 years now, but the exercises also include the latest ones like the New Horizons and the Juno Space Probe. In the case of space telescopes, similarly to terrestrial radio telescopes that are not included in traditional problems, an inescapable technical issue is improving resolution. In today's world of electronic devices, it is important to observe the Sun and solar flares as well, so the spacecraft designed for this purpose is also included in our problems.

Another group of problems of the chapter focus on asteroid research and its results that have become increasingly important. The favourite topics of the media and film industry include cosmic disasters. Since news about planetary asteroids passing in the vicinity of Earth often give rise to unjustifiable fear (since most of them pass outside the Moon's orbit), common sense can be improved by problems using the probability method. Nevertheless, such events (such as the Chelyabinsk meteor impact) actually take place, so many astronomers have the primary task of mapping asteroids passing close to Earth. There are several tasks to deal with asteroids, with the consequences of their possible impact. We show an example for an existing impact crater and discuss the issue of preventing possible impacts by using images of an object shot into the nucleus of a comet.

Finally, a separate chapter was devoted to problems that promote inquiry-based learning and independent student activity. These problems therefore provide new opportunities in the field of methodology, not in the field of the aforementioned astronomical results.

Acknowledgements

We would like to thank our teacher, Dr. Péter Tasnádi, for helping our work with many helpful advice.

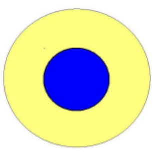
1. Introduction of exoplanets
DIMENSIONS AND DISTANCES

1.1 Most sci-fi stories are based on the premise that the problem of fast interstellar travel has been solved. The 40-year anniversary of the Voyager-1 space probe gives topicality to this issue. However, the technologies currently known are predicting very long travel times.
As we discover more and more exoplanets, sooner or later we will find one in the vicinity that shows signs of life. At this point, there will soon be a demand for manned or unmanned space missions to study the extra-terrestrial life form.
The following table contains the data of a few exoplanets, including their distance from us.
(a) So far, the fastest unmanned spacecraft was *New Horizons* (v = 58,320 km/h), while among the man-driven ones Apollo-10's speed was the highest (v = 39,896 km/h). Find the time required to reach the closest exoplanet in the table at these speeds.

(b) Engineers believe it is possible to build a spacecraft whose speed can exceed 1000 km/s. Find the time required for the journey to the close exoplanets listed in the table with such a spacecraft.

Name	Constellation	Distance (light-year)	Distance from the star (AU)	Mass (Earth's mass)	Orbital period
Epsilon Eridani b	Eridanus	10.5	3.4	500	6.9 years
Gliese-581g	Libra	20.3	0.14	3.1	36 days
Giese-674	Ara	14.8	0.04	12	4.7 days
Gliese-876d	Aquarius	15	0.02	8	1.9 days
Gliese-832b	Grus	16.1	3.4	200	9.3 years
Gliese-176	Taurus	31	0.07	25	8.7 days
Fomalhaut b	Piscis Austrinus	25	115	600	872 years
61 Virginis b	Virgo	28	0.05	5.1	4.2 days

1.2 At first, astronomers discovered huge Jupiter-sized exoplanets, but as the measurement technologies evolved, so-called super-Earths were also found, which were "only" a few times larger than Earth. (Nowadays we already know exoplanets smaller than Earth, e.g. Kepler-42d.)
A super-Earth is not necessarily similar to Earth. It may be a gas giant similar to Jupiter, it may be an icy world like Uranus or Neptune, but it can be a rocky planet like the inner planets of the Solar System. To find out the type, the internal structure and the atmospheric composition of a discovered exoplanet, it needs to be investigated thoroughly.
Scientists first determine the mass and the size (volume) of the exoplanet. Knowing the mass, the volume and thus the density, they can model its inner composition and structure well enough. The creation of the model starts by selecting a suitable core-shell model. (This is not a definitive definition of the inner structure and the composition of the celestial body, it is only a starting point for further investigations.)
(a) According to a simple model of the inner structure of exoplanets, suppose our imaginary exoplanet has a spherical, solid rock core and its outer shell is a thick layer of ice. Find the radius of the exoplanet if the volume of the core and the shell is $4.18 \cdot 10^{12}\,\mathrm{km}^3$ and $2.92 \cdot 10^{13}\,\mathrm{km}^3$, respectively.



(b) Earth's volume is $1.1 \cdot 10^{12}\,\mathrm{km}^3$. How many times greater is the volume of the core and of the shell of the exoplanet than Earth's volume?

(c) Suppose that the astronomers who discover the super-Earth can determine its mass as well and find that it is 8.3 times Earth's mass. Find the mass and the density of the exoplanet.

(d) Based on the distance of the exoplanet from its star, it is assumed that the thick shell consists of solid ice, whose density is 900 kg/m^3. Find the density of the core of the exoplanet.

1.3 With the help of the Hubble Space Telescope it was proven that exoplanet HD 209458b (also known as Osiris) located in the Pegasus constellation at a distance of 150 lightyears orbits around its star whose mass is similar to the Sun with a period of 3.5 days. The gas giant orbits around its star well within the radius of the orbit of Mercury, only at a distance of 0.047 AU.
Hubble's COS spectrograph found water and heavy elements, carbon and silicon in the atmosphere of the exoplanet, whose temperature is several thousand degrees. Measurements based on the analysis of hydrogen, carbon and silicon spectrum lines have shown a strong stellar wind that drives the blown-off gas away like the train of a comet. That is, the planet is orbiting so close to its star that its radiation blows the atmosphere of the planet into space.

The matter loss of the atmosphere of the exoplanet is $4 \cdot 10^{11}$ g/s. Find the mass of matter lost by the exoplanet
(a) in one day,

(b) in one year.

(c) Jupiter's mass is $1.9 \cdot 10^{27}\,\mathrm{kg}$, its radius is $7.13 \cdot 10^{7}\,\mathrm{m}$. HD 209458b's mass is approximately 70% of Jupiter's mass and its radius is approximately 1.4 times Jupiter's radius. Find Jupiter's and exoplanet HD209458b's density.

(d) Assume that similarly to Jupiter, this exoplanet also has a rocky core whose mass is 18 times Earth's mass. Find the mass of HD 209458b's atmosphere.

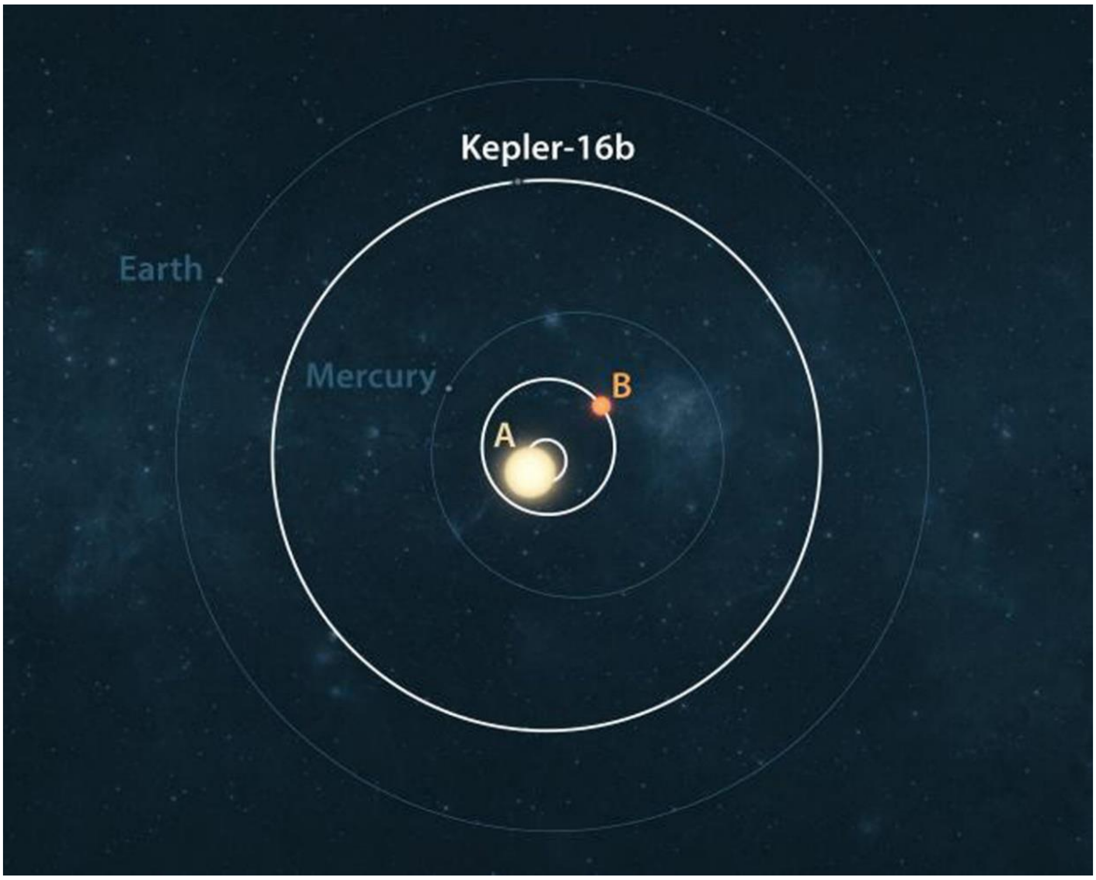
(e) How long will it take until HD 209458b completely loses its atmosphere if the rate of matter loss remains the same?

1.4 From planet Kepler-16b discovered by NASA's Kepler space telescope two suns can be seen in the sky instead of one. Of the planets discovered in our Galaxy so far, Kepler-16b resembles most to planet Tatooine, Luke Skywalker's home planet known from the popular science fiction movie Star Wars. In reality Kepler-16b is not habitable, because it does not have a cold, solid surface, but like Tatooine, it orbits around two stars.

Luke Skywalker is watching the twin sunset.

The binary star (the greater is Kepler-16A and the smaller is Kepler-16B) is at a distance of 200 light-years in the direction of the Cygnus constellation. The smaller star, Kepler-16B orbits around its heavier

twin on a circular orbit whose radius is 0.2 AU = 30 million km. The diameter of the two stars is 890000 km and 300000 km, respectively.
Exoplanet Kepler-16b orbits around Kepler-16A at a distance of 0.7 AU = 105 million km. Its orbit is nearly a perfect circle.



https://www.jpl.nasa.gov/spaceimages

(a) Imagine that you are standing on the surface of "Tatooine" and watching the two stars in the sky like Luke Skywalker was watching the twin suns at sunset. Determine the maximum possible angular separation of the two stars when watched from Tatooine.

(b) Determine the angle in which the diameter of the stars is seen from Tatooine at the time when they are furthest apart in the sky.

1. Introduction to exoplanets

THE HABITABLE ZONE OF EXOPLANETS, MODELLING THE HABITABLE ZONE

1.5 Around a star, the habitable zone means the zone where water can exist in liquid state permanently on the surface of a rocky planet, that is, (under conditions similar to Earth) if the temperature is between 273 K and 373 K. The position of this zone depends on the energy output of the star. The brighter the star, the more energy it transfers to its planet, so around a star with higher luminosity the surface temperature of a planet is also higher.

The next table contains the surface temperature of planets in kelvins as the function of the luminosity of the star (*L*) and the distance between the planet and the star (*d*). The luminosity of the star is given relative to the Sun's luminosity and the distance is given in Astronomical Units.
The model used for calculating temperature assumes that the albedo (reflexive property) of the exoplanet is similar to that of Earth and also that the concentration of carbon dioxide is the same as on Earth.

L**d	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**	**2.2**	**2.4**	**2.6**	**2.8**	**3**	**3.2**
0.1	361	255	209	181	162	147	136	126	120	114	109	104	100	97	93	90
0.5	540	382	312	270	242	220	204	191	180	171	163	156	150	144	139	135
1	642	454	371	321	287	262	243	227	214	203	194	185	178	172	166	161
1.5	711	503	410	355	318	290	269	251	237	225	214	205	197	190	184	178
2	764	540	441	382	342	312	289	270	255	242	230	220	212	204	197	191
2.5	808	571	466	404	361	330	305	286	269	255	243	233	224	216	209	202
3	845	598	488	423	378	345	319	299	282	267	255	244	234	226	218	211
3.5	878	621	507	439	393	359	332	311	293	278	265	254	244	235	227	220
4	908	642	524	454	406	371	343	321	303	287	274	262	252	243	235	227
4.5	935	661	540	468	418	382	354	331	312	296	282	270	259	250	242	234
5	960	679	554	480	429	392	363	340	320	304	290	277	266	257	248	240
5.5	983	695	568	492	440	402	372	348	328	311	297	284	273	263	254	246
6	1005	711	580	503	450	410	380	355	335	318	303	290	279	269	260	251
6.5	1025	725	592	513	459	419	388	363	342	324	309	296	284	274	265	256
7	1045	739	603	522	467	426	395	369	348	330	315	302	290	279	270	261

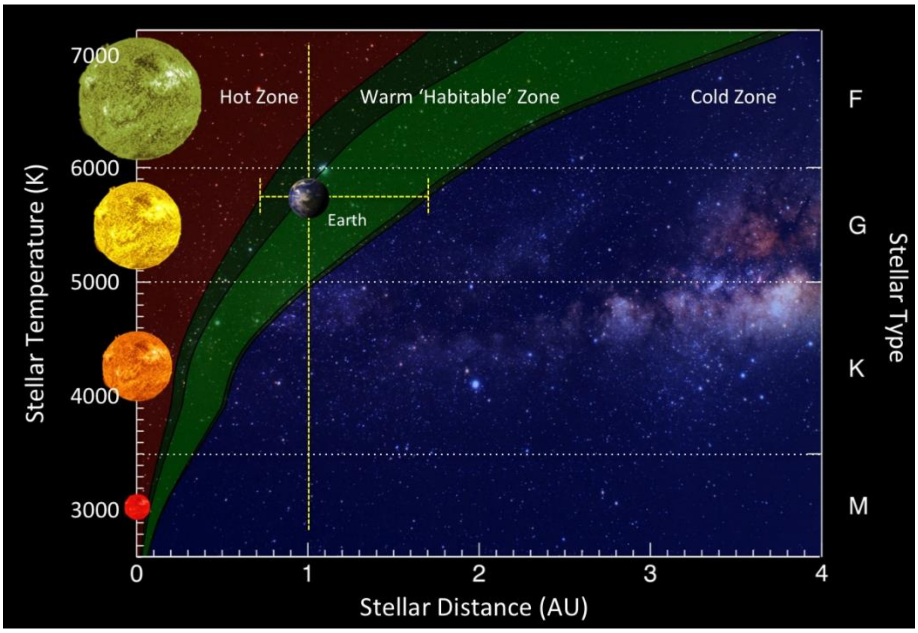
(a) For each luminosity value, colour the temperatures that are close to or within the interval where water is liquid.

(b) How does the habitable zone change as the luminosity of the star increases?

(c) According to this model, which planet(s) of the Solar System is (are) in the habitable zone?

(d) Calculate the area of the Sun's (L = 1) habitable zone in this model.

Remark:
The figure shows the habitable zone of stars of different temperature, therefore different luminosity as a function of the distance from the star. The green part shows the habitable zone. According to some models the dark blue zones on the outer border of the green zone also belong into the habitable zone. Exoplanets in the claret zone are too hot while exoplanets in the blue zone are too cold to hold water in liquid state permanently. Earth is also shown close to the inner border of the habitable zone.



http://spacemath.gsfc.nasa.gov

1.6 Astronomers have discovered almost 4,000 exoplanets around the nearby stars. When they discover one, they also investigate whether liquid water can be present on its surface. Planets that orbit in the zone called habitable zone by astronomers have temperature conditions where water can exist in liquid state permanently on the surface (under conditions similar to Earth). In our Solar System Mercury and Venus are too close to the Sun, their surface is so hot that water cannot remain in liquid state on them, it would evaporate from their surface. On Mars and planets beyond it water freezes to ice. Our Earth orbits in the Sun's habitable zone.
According to a simple model, water freezes on the surface of a planet that orbits at distance D given in astronomical units around a star whose mass is given as M times the Sun's mass, if

$$M - 0.8D \leq 0.12.$$

(a) Draw the solution of the inequality in a coordinate system. On the vertical axis show the mass of the star as a multiple of the Sun's mass in the interval [0.0; 2.0] and on the horizontal axis the distance of the planet from the star in Astronomical Units in the interval [0.0; 3.0] using a step of one tenth. Colour the region under the freezing point green.

(b) Water boils on the surface of the planet if

$$M - 1.2D \geq 0.18.$$

Colour the region above the boiling point blue.

(c) In addition to too high or too low temperatures, captured rotation can also make a planet uninhabitable. Captured rotation means that the orbital period and the period of rotation of a celestial body are equal. In such a case, the planet always turns the same half to its star, so this side always has daylight (and heat), while on the other side eternal night darkness and the resulting very low temperature are probable. For planets with captured rotation that have an atmosphere, high-speed winds may reduce the temperature difference between the two sides, but the planet still does not become habitable.
An exoplanet orbits around its star with captured rotation if

$$M - 3.3D \geq -1.3$$

Colour the region where the planet has captured rotation red. What can be stated about the uncoloured region?

(d) Which of these imaginary exoplanets orbits in the habitable zone of its star?

Lehel : D = 2.0 AU, M = 1M_{Sun}
Hades : D = 0.5 AU, M = 2M_{Sun}
Oceania: D = 2.0 AU, M = 2M_{Sun}

Remarks:

1. Based on the measurements of the Kepler telescope we believe that among the hundred thousand million stars of the Milky Way those are rare that have no planets, so even a thousand million exoplanets can exist only in our Galaxy, and among these many can be habitable. These exoplanets are the primary objectives of the astrobiological research outside the Solar System, we are searching for life similar to life on Earth on these.

2. In the English terminology exoplanets that orbit in the habitable zone are called "Goldilocks planets". The name refers to the small girl from the fairy tale *Goldilocks and the Three Bears*, who tasted the porridge of the bears at the home of a bear family. She found the father's porridge too hot, the mother's porridge too cold and the baby bear's porridge just right, neither too hot nor too cold.

1.7 To have a liquid water on a planet's surface permanently, the exoplanet must be close enough to its star to melt the ice, but not too close to vaporize the water on its surface. The suitable zone is called habitable zone.

Based on a simple model, the temperature of an exoplanet can be calculated with the following formula:

$$T = 0.6 \cdot T_{\mathrm{eff}} \cdot \sqrt{\frac{R}{d}}$$

where T_{eff} is the effective temperature of the star in kelvins, R is the radius of the star, d is the average distance of the exoplanet from its star.

(a) If a star has the same temperature and radius as the Sun (T_{eff} = 5,770 K and R = 700,000 km), find the distance limits between which the temperature is suitable for an exoplanet to have liquid water on its surface. Assume that the conditions on the planet are similar to those on Earth.

(b) Is Earth habitable based on these results? Why?

(c) Canopus (α Carinae), which is visible from the southern hemisphere, is the second brightest star in the sky. Its temperature is 7,000 K and its radius is 70 times the radius of the Sun. Assuming Earth-like conditions, find the range in which water can be in liquid state on the surface of an exoplanet around this star. Compare the position of the calculated zone with the orbital radii of the planets of the Solar System.

1.8 (*Problem from a preparatory course for Students' Olympiad*)

An exoplanet orbits on an orbit whose eccentricity is e = 0.2 and semi-major axis is a = 4 AU. The habitable zone of the star is between r_{inner} = 3 AU and r_{outer} = 4 AU. Calculate the percent of the orbital period that the star spends in the habitable zone.

1. Introduction of exoplanets

KEPLER'S THIRD LAW AND THE APPLICATION OF THE LAW OF GRAVITY FOR EXOPLANETS

1.9 Star CoRoT-2a is 880 light-years from us in the direction of the Aquila constellation. Its planet named CoRoT-2b is a gas planet 1.4 times greater than Jupiter, and its mass is three times the mass of Jupiter. The planet orbits around the star with a period of 1.7 days, at a distance of only 5 million km, so close that its upper atmosphere is heated to 1,500 K by the radiation of the star.
(a) Find the mass of star CoRoT-2a in terms of the Sun's mass. Compare exoplanet CoRoT-2b's orbital radius with the orbital radii of the planets of the Solar System.

(b) According to a simple model the mass of the atmosphere of a planet is 50% of the mass of the planet. Jupiter's mass is 315 times Earth's mass, $1.9 \cdot 10^{27}$ kg. CoRoT-2b loses approximately 5 million tons of matter per second due to the radiation of its star. If we assume that this rate of evaporation remains constant as given above, how long will it take until the planet loses its atmosphere completely?

1.10 (*A problem from the International Astronomy and Astrophysics Olympiad, 2015.*)
A few exoplanets have been discovered around star GJ 876 ($M_{GJ876} = (0.33 \pm 0.03)M_{Sun}$), their data are given in the following table where M_S is the Sun's mass, m_E is Earth's mass and m_J is Jupiter's mass ($m_J = 1.8913 \cdot 10^{27}$ kg). Assume that the exoplanets orbit in the same direction around star GJ 876. Two planets are said to be in resonance if the ratio of the orbital periods can be approximated well with the ratio of two small integers. Find exoplanets in orbital resonance in the GJ 876 system.

GJ 876 system	**Mass**	**Semi-major axis (AU)**
GJ 876b	2.276 m_J	0.2083
GJ 876c	0.714 m_J	0.1296
GJ 876d	6.8 m_E	0.0208
GJ 876e	15 m_E	0.334

1.11 The nearby red dwarf Gliese-581 is 20 light-years from us in the direction of the Libra constellation. Its "crowded" system has six planets. The mass of exoplanet Gliese-581g is only three times Earth's mass, so it is more likely to be a rocky planet than a gas giant. The planet orbits around its star with a period of 37 days with captured rotation, that is, it always faces its star with the same side: on one of the hemispheres there is eternal day, on the other one eternal night.

(a) Draw a model of the Gliese-581 planet system. In the drawing 1 cm should correspond to 0.01 AU and in the case of the radius of exoplanets 1 mm should correspond to 5,000 km.
Use the data of the following table:

Planet	**Year of discovery**	**Distance from the star (AU)**	**Orbital period (day)**	**Diameter (km)**
Gliese-581b	2005	0.041	5.37	50,000
Gliese-581c	2007	0.072	12.9	20,000
Gliese-581d	2007	0.22	66.9	25,000
Gliese-581e	2009	0.028	3.15	15,000
Gliese-581f	2010	0.76	433	25,000
Gliese-581g	2010	0.15	36.6	20,000

(b) Determine the mass of Gliese 581.

1.12 Exoplanet Kepler-10b, an Earth-sized exoplanet that orbits around star Kepler-10 with an orbital period of 20 hours at a distance of 2.5 million kilometres, was discovered in 2010 with the Kepler Space Telescope in the direction of the Dragon constellation, at a distance of 560 light years from us. The surface temperature of the planet exceeds 1,500 kelvins. By studying the transit of the exoplanet (its passing in front of its star and the resulting decrease in luminosity), the size of Kepler-10b is estimated to be 1.4 times the Earth's size. Its average density is 8.8 g/cm^3, which is higher than the density of iron and three times the density of the silicon-rich layer on Earth's surface.

(a) Find the mass of Kepler-10b.

(b) Find the value of gravitational acceleration on exoplanet Kepler-10b.

(c) Find the weight of a 60-kg human on Earth and on exoplanet Kepler-10b.

Solutions 1.

1.1 (a) Of the exoplanets listed in the table the closest one is Epsilon Eridani b, at a distance of 10.5 light years.

10.5 light-year $= 10.5 \cdot 9.46 \cdot 10^{12} = 9.9 \cdot 10^{13}$ km

At the speed of the unmanned spacecraft we would reach exoplanet Epsilon Eridani b in

$$t_1 = \frac{9.9 \cdot 10^{13}}{58320} = 1.7 \cdot 10^6 \,\text{h} \approx 190000 \text{ years}.$$

At the speed of the human-driven spacecraft we would reach it in

$$t_2 = \frac{9.9 \cdot 10^{13}}{39896} = 2.5 \cdot 10^9 \,\text{h} \approx 280000 \text{ years}.$$

(b) The speed of a spacecraft that can travel at 1,000 km/s is 1/300 of the speed of light, so it covers a distance of one light-year in 300 years.

The following table contains the travel times calculated with the speed of 1,000 km/s.

Name	**Distance (light-year)**	**Travel time (year)**
ε Eridani b	10.5	3150
Gliese-581g	20.3	6090
Giese-674	14.8	4440
Gliese-876d	15	4500
Gliese-832b	16.1	4830
Gliese-176	31	9300
Fomalhaut b	25	7500
61 Virginis b	28	8400

Remark:
An exoplanet has also been found around star Proxima Centauri, which belongs to the trinary star system closest to us, Alfa Centauri. This exoplanet is in the habitable zone. The travel time to this Earth-sized exoplanet, which is 4.3 light-years from us, would still take approximately 1300 years.

1.2 (a) The total volume is

$$V = 4.18 \cdot 10^{12} + 2.92 \cdot 10^{13} = 3.34 \cdot 10^{13} \,\text{km}^3$$

Assuming that it is spherical,

$$\frac{4}{3}\pi \cdot R^3 = 3.34 \cdot 10^{13} \,\text{km}^3$$

$$R = 2.00 \cdot 10^4 = 20000 \text{ km}$$

(b) $\dfrac{V_{\text{core}}}{V_{\text{Earth}}} = \dfrac{4.18 \cdot 10^{12}}{1.1 \cdot 10^{12}} = 3.8 \approx 4$

The volume of the core of the exoplanet is approximately four times that of Earth.

$$\frac{V_{shell}}{V_{Earth}} = \frac{2.92 \cdot 10^{13}}{1.1 \cdot 10^{12}} = 26.5 \approx 30$$

The volume of the shell is almost 30 times that of Earth.

(c) The mass of the exoplanet is

$$m = 8.3 \cdot 5.9 \cdot 10^{24} = 4.9 \cdot 10^{25} \,\text{kg}$$

The density of the exoplanet is

$$\rho = \frac{m}{V} = \frac{4.9 \cdot 10^{25}}{3.34 \cdot 10^{22}} = 1500 \,\frac{\text{kg}}{\text{m}^3}$$

Remark:
Although the density 1500 kg/m^3 is not much higher than the density of ice, the exoplanet still has a large core whose density is much higher.

(d) $m_{\text{core}} = m - m_{\text{shell}} = m - V_{\text{shell}} \cdot \rho_{\text{shell}} =$

$= 4.9 \cdot 10^{25} - 2.92 \cdot 10^{13} \cdot 10^9 \cdot 900 =$

$= 2.4 \cdot 10^{25}$ kg.

The density of the core of the exoplanet is

$$\rho_{\text{core}} = \frac{m_{\text{core}}}{V_{\text{core}}} = \frac{2.4 \cdot 10^{25}}{4.18 \cdot 10^{12} \cdot 10^9} = 5700 \,\frac{\text{kg}}{\text{m}^3}$$

This density suggests a rocky core.

1.3 (a) In 1 day = 86400 s exoplanet HD 209458b loses

$$4 \cdot 10^{11} \,\frac{g}{s} \cdot 86400 = 3.5 \cdot 10^{16} \,\text{g} = 3.5 \cdot 10^{10} \,\text{t}$$

of matter.

(b) In one year the matter loss is

$$3.5 \cdot 10^{10} \cdot 365 = 1.3 \cdot 10^{13} \,\text{t}.$$

(c) The planets are nearly spherical, so Jupiter's volume is

$$V_{\text{J}} = \frac{4}{3}\pi \cdot \left(7.13 \cdot 10^7\right)^3 = 1.5 \cdot 10^{24} \,\text{m}^3$$

Jupiter's density is

$$\rho_{\text{J}} = \frac{m_{\text{J}}}{V_{\text{J}}} = \frac{1.9 \cdot 10^{27}}{1.5 \cdot 10^{24}} = 1267 \,\frac{\text{kg}}{\text{m}^3}$$

The density of the exoplanet is

$$\rho = \frac{0.7 M_{\text{J}}}{(1.4)^3 V_{\text{J}}} = 323 \,\frac{\text{kg}}{\text{m}^3}$$

Remark:
HD 209458b is an exoplanet whose density is among the smallest ones.

(d) The mass of the exoplanet is

$$m = 0.7 \cdot M_{\mathrm{J}} = 1.3 \cdot 10^{27}\,\mathrm{kg}$$

The mass of the atmosphere is

$$m_{\mathrm{atmosphere}} = m - 18 \cdot M_{\mathrm{E}} =$$

$$1.3 \cdot 10^{27} - 18 \cdot \left(5.9 \cdot 10^{24}\right) = 1.2 \cdot 10^{27}\,\mathrm{kg}$$

(e) $t = \dfrac{1.2 \cdot 10^{27}}{1.3 \cdot 10^{16}} = 9.2 \cdot 10^{11}$ years

Remark:
The rate of matter loss of the exoplanet seems to be huge but the received result, 920 thousand million years is almost 70 times the age of the Universe. With a few exceptions planets can live longer than their stars (lifespan of about 10–20 thousand million years) despite such huge matter loss.

1.4 (a) In the investigated position the two stars and the exoplanet form the vertices of a right-angled triangle whose hypotenuse is the distance between the heavier star and the exoplanet and one of the perpendicular arms is the distance between the two stars:

$$\sin\phi = \frac{30}{105} = 0.286\text{, so } \varphi = 17°.$$

(b) Kepler-16A's diameter is 890,000 km and it is 105 million kilometres from the exoplanet. So from "Tatooine" the diameter is seen in an angle

$$\alpha = \frac{0.89}{105} = 0.0085\ \mathrm{rad} = 0.49°.$$

(c) Using Pythagoras' theorem the distance of Kepler-16B is

$d^2 = 105^2 - 30^2$, so
d = 101 million km.

$$\beta = \frac{0.30}{101} = 0.0030\ \ \mathrm{rad} = 0.17°$$

By comparison: from Earth, Sun and Moon are seen in an angle of approximately 0.5°, the apparent size of Kepler-16A is approximately the same, and of Kepler-16B much smaller.

1.5 (a) The table shows a correct colouring. (In a few cells the freezing point and the boiling point of water can be rounded down or up.)

(b) As the luminosity of stars increases, the habitable zone gets further from them and its width also increases.

(c) Venus and Earth
The simplicity of the model is reflected by the fact that it considers Venus habitable, because it does not take greenhouse effect into account.

(d) The requested region is an annular ring with area

$$t = \pi \cdot \left(r_{\mathrm{o}}^2 - r_{\mathrm{i}}^2\right)$$

where r_{o} is the outer radius of the habitable zone and r_{i} is the inner radius.
From the values coloured in row $L = 1$ of the table

r_{i} = 0.6 AU and
r_{o}=1.2 AU

so the requested area is

$$t = \pi \cdot \left(1.2^2 - 0.6^2\right)^2 = \ 7.6 \cdot 10^{12}\,\mathrm{km}^2$$

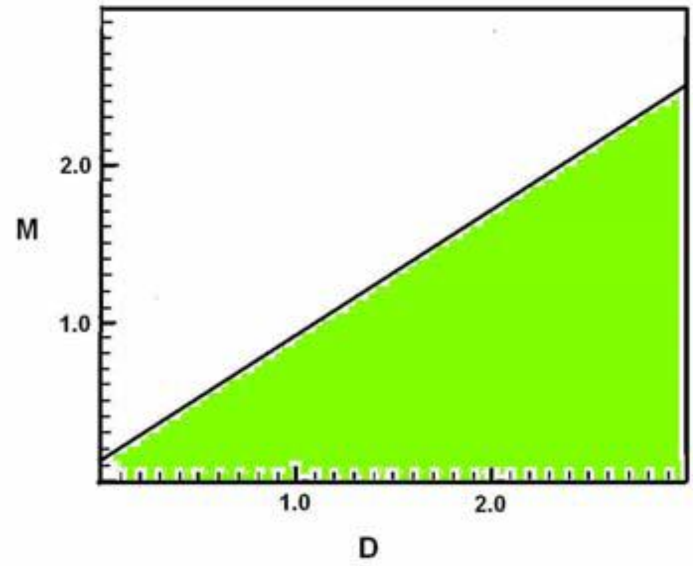
Remark:
Looking more closely at the outer radius of the habitable zone, value 1.1 AU is a better approximation of the freezing point of water (273 K). Using this value

$$t = \pi \cdot \left(1.1^2 - 0.6^2\right)^2 =$$

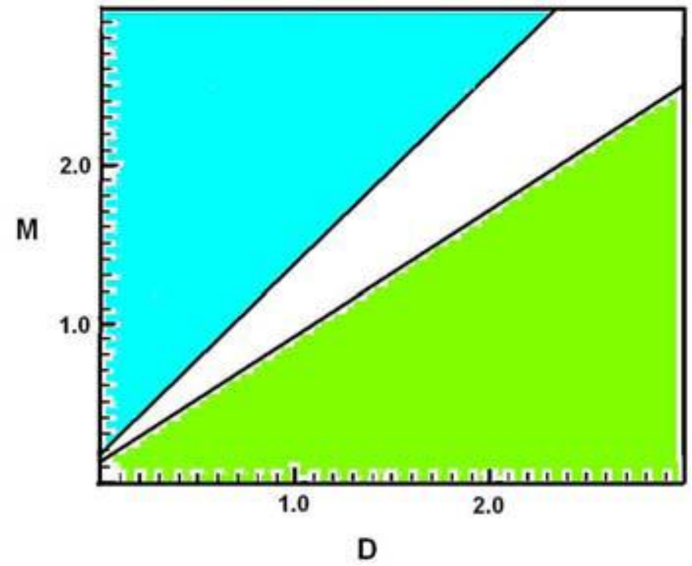
$$= 2.7 \cdot \left(150000000\right)^2 = 6 \cdot 10^{12}\,\mathrm{km}^2$$

L\d	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**	**2.2**	**2.4**	**2.6**	**2.8**	**3**	**3.2**
0.1	361	255	209	181	162	147	136	126	120	114	109	104	100	97	93	90
0.5	540	382	312	270	242	220	204	191	180	171	163	156	150	144	139	135
1	642	454	371	321	287	262	243	227	214	203	194	185	178	172	166	161
1.5	711	503	410	355	318	290	269	251	237	225	214	205	197	190	184	178
2	764	540	441	382	342	312	289	270	255	242	230	220	212	204	197	191
2.5	808	571	466	404	361	330	305	286	269	255	243	233	224	216	209	202
3	845	598	488	423	378	345	319	299	282	267	255	244	234	226	218	211
3.5	878	621	507	439	393	359	332	311	293	278	265	254	244	235	227	220
4	908	642	524	454	406	371	343	321	303	287	274	262	252	243	235	227
4.5	935	661	540	468	418	382	354	331	312	296	282	270	259	250	242	234
5	960	679	554	480	429	392	363	340	320	304	290	277	266	257	248	240
5.5	983	695	568	492	440	402	372	348	328	311	297	284	273	263	254	246
6	1005	711	580	503	450	410	380	355	335	318	303	290	279	269	260	251
6.5	1025	725	592	513	459	419	388	363	342	324	309	296	284	274	265	256
7	1045	739	603	522	467	426	395	369	348	330	315	302	290	279	270	261

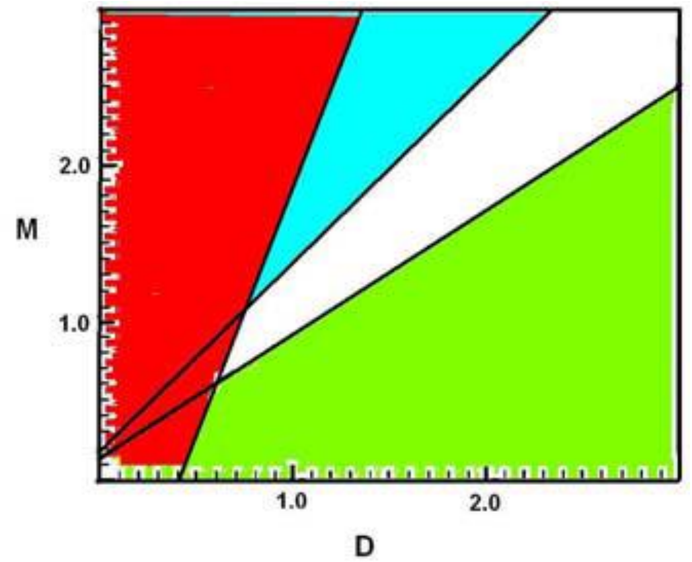
1.6 (a)



(b)



(c)



The uncoloured (white) region shows the habitable zone.

(d) Lehel:

$M-0.8D=1-0.8\cdot 2=-0.6\leq 0.12$

Water freezes, it is not in the habitable zone.

Hades:

$M-1.2D=2-1.2\cdot 0.5=1.4\geq 0.18$

Water boils away, it is not in the habitable zone. (moreover,

$M-3.3D=2-3.3\cdot 0.5=0.35\geq -1.3$,

so it has captured rotation.)

Oceania:

$M-0.8D=2-0.8\cdot 2=0.4\leq 0.12$,

$M-1.2D=2-1.2\cdot 2=-0.4\leq 0.18$

$M-3.3D=2-3.3\cdot 2=-4.6\leq -1.3$

none of the inequalities holds.

Oceania is in the habitable zone of its star. It orbits around it far enough to have a different orbital and rotational period, so on the planet days and nights change each other. The surface temperature is also suitable for the permanent existence of liquid water.

1.7 (a) The boiling point of water (under Earth-like conditions) is 373 K:

$$373K=0.6\cdot 5770\cdot\sqrt{\frac{700000}{d_{\text{inner}}}}$$

$d_{\text{inner}}=6.0\cdot 10^{7}=60$ million km

The melting point of ice (under Earth-like conditions) 273 K:

$$273K=0.6\cdot 5770\cdot\sqrt{\frac{700000}{d_{\text{outer}}}}$$

$d_{\text{outer}}=1.1\cdot 10^{8}=110$ million km

Water can exist in liquid state on planets that orbit at a distance not smaller than 60 million km and not greater than 110 million km from the star.

(b) No. The model forming the basis of the formula does not take the effect of the atmosphere into account.

(c) $T = 0.6 \cdot 7200 \cdot \sqrt{\frac{70 \cdot 700000}{d}}$

$373K = 0.6 \cdot 7000 \cdot \sqrt{\frac{70 \cdot 700000}{d_{\mathrm{inner}}}}$

$d_{\mathrm{inner}} = 6.2 \cdot 10^9\,\mathrm{km} = 41\ \mathrm{AU}$

This is approximately Pluto's mean distance from the Sun.

$273K = 0.6 \cdot 7200 \cdot \sqrt{\frac{30 \cdot 700000}{d_{\mathrm{outer}}}}$

$d_{\mathrm{outer}} = 1.2 \cdot 10^{10}\,\mathrm{km} = 77\,\mathrm{AU}$

More than twice Neptune's orbital radius (30 AU). Based on these data Canopus' habitable zone would be outside the orbit of each planet of the Solar System.

1.8 $c = 0.2a = 0.8\ \mathrm{AU}$

$a - c = 3.2 > 3\ \mathrm{AU}$,

so the planet is never inside the inner boundary of the habitable zone.

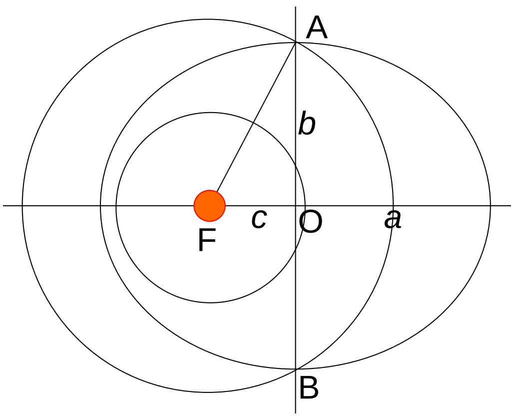
The distance of the outer boundary of the habitable zone is equal to the semi-major axis, so the planet crosses the outer boundary of the habitable zone at the endpoints of the semi-minor axis.

The areal velocity of the planet is constant, so we need to calculate what percent of the area of the ellipse is the area of the smaller sector bounded by radii FA and FB.

The area of the sector is the area of the half ellipse minus twice the area of triangle OFA:

$$\frac{\frac{ab\pi}{2} - bc}{ab\pi} = \frac{\frac{\pi}{2} - \frac{c}{a}}{\pi} = \frac{1}{2} - \frac{c}{a\pi} = \frac{1}{2} - \frac{e}{\pi} =$$

$$= 0.5 - \frac{0.2}{\pi} = 0.44 = 44\%$$



1.9 (c) $M = \frac{4\pi^2 r^3}{\gamma T^2} =$

$$= \frac{4\pi^2 (5 \cdot 10^9)^3}{6.7 \cdot 10^{-11} \cdot (1.7 \cdot 24 \cdot 3600)^2} =$$

$$= 3.4 \cdot 10^{30}\,\mathrm{kg}$$

Approximately one and a half times the Sun's mass, so it is a star similar to the Sun.

The exoplanet orbits well inside Mercury's orbit.

(b) $m_{\mathrm{atmosphere}} = 0.5 \cdot 3 \cdot 1.9 \cdot 10^{27} = 2.9 \cdot 10^{27}\,\mathrm{kg}$

$1\ \mathrm{year} \approx 3.1 \cdot 10^7\,\mathrm{s}$.

In one year the planet loses $5 \cdot 10^9 \cdot 3.1 \cdot 10^7 = 1.6 \cdot 10^{17}\,\mathrm{kg}$ of matter, so it loses its atmosphere completely in

$$t = \frac{2.9 \cdot 10^{27}}{1.6 \cdot 10^{17}} = 1.8 \cdot 10^{10}\ \mathrm{years},$$

that is, in 18 thousand million years.

Remark:
The rate of matter loss of the exoplanet seems to be huge but the received result, 18 thousand million years is more than the age of the Universe. With a few exceptions planets can live longer than their stars (lifespan of about 10–20 thousand million years) despite such huge matter loss.

1.10 Kepler's third law states the relationship between the orbital period (T) and the length of the semi-major axis (a) of the orbit of a(n) (exo)planet (of mass m_b) orbiting around a star (of mass M_s) is

$$\frac{a^3}{T^2} = \frac{\gamma(M_s + m_b)}{4\pi^2},$$

If two exoplanets have masses m_1 and m_2, then the ratio of the squares of the orbital periods is

$$\frac{T_1^2}{T_2^2} = \frac{(M_s + m_2)a_1^3}{(M_s + m_1)a_2^3}$$

It is reasonable to relate masses to the same celestial body, for example Jupiter (although the concrete values can be calculated because Jupiter's mass is given, they are not important for the exercise). The Sun's mass is 1047 times Jupiter's mass, m_J and Jupiter's mass is 318 times Earth's mass, m_F.

The mass of star GJ 876 is therefore

$M_{\mathrm{GJ876}} = (0.33 \pm 0.03) \cdot M_{\mathrm{Sun}} =$
$= (345.5 \pm 31) \cdot m_{\mathrm{J}}$

The mass of exoplanet GJ 876d (hereinafter simply d) is

$m_{\mathrm{d}} = 6.8 m_{\mathrm{E}} = 0.0214 m_{\mathrm{J}}$,

and the mass of exoplanet e is

$m_e = 15m_E = 0{,}047m_J$,

The combined mass of the exoplanet and the star can be approximated well with the mass of the star, so our formula simplifies to the form known by high school students:

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

The known exoplanets of star GJ 876 follow each other in the following order of increasing distance from the star: *d*, *c*, *b* and *e*. This order is established from the semi-major axes of the exoplanet orbits. (The exoplanets receive the small letters after the name of the star in the order of their discovery, starting from letter *b*.)

The ratio of the orbital periods of exoplanets *c* and *d* is

$$\frac{T_c}{T_d} = \sqrt{\frac{a_c^3}{a_d^3}} \approx 31.68,$$

which cannot be written as the ratio of small integers.

If the orbital period of exoplanet *b* or *e* is related to the orbital period of *d*, an even higher number is acquired, so these ratios are not worth calculating.

Similarly to the above, the ratio of the orbital periods of *b* and *c* is

$$\frac{T_b}{T_c} = \sqrt{\frac{a_b^3}{a_c^3}} \approx 2.04,$$

which can be approximated with a resonance of ratio 2:1.

We also have to investigate the ratio of the orbital periods of exoplanets *e* and *b*:

$$\frac{T_e}{T_b} = \sqrt{\frac{a_e^3}{a_b^3}} \approx 2.03,$$

which also corresponds to a resonance of 2:1, similarly to the previous one.

From these two resonances it follows that exoplanets *e* and *c* are also in resonance with a ratio 4:1.

So in the planetary system of star GJ 876 three planets are in resonance (T_c:T_b:T_e = 1:2:4). For checking, the measured orbital periods (given in the exoplanet catalogue) are 30.23 days, 61.03 days and 124.69 nap, which agree with our results.

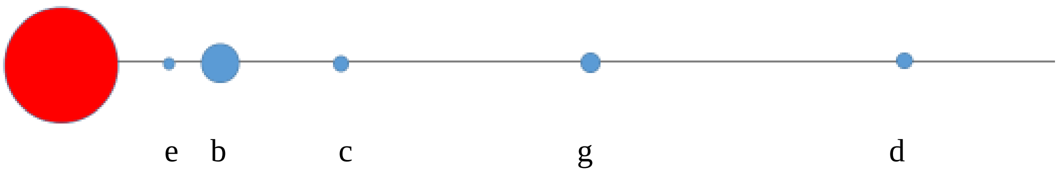
Such resonance can also be found in our Solar System, albeit not between planets, but between Jupiter's three inner Galilean moons (T_{Io}:T_{Europa}:$T_{Ganymede}$ = 1:2:4, their orbital periods are 1.77 days, 3.55 days and 7.16 days, respectively).

1.11 (a) The following table contains the data to be drawn:

Planet	**Distance (cm)**	**Diameter (mm)**
Gliese-581b	4	10
Gliese-581c	7	4
Gliese-581d	22	5
Gliese-581e	3	3
Gliese-581f	76	5
Gliese-581g	15	4

With the requested scale Gliese-581f would not fit on many paper sizes, so either we do not draw it or we choose another scale based on the size of the available paper.

The next figure shows the planetary system Gliese-581 to scale except for planet *f*. The star is on the left, at the beginning of the ray.



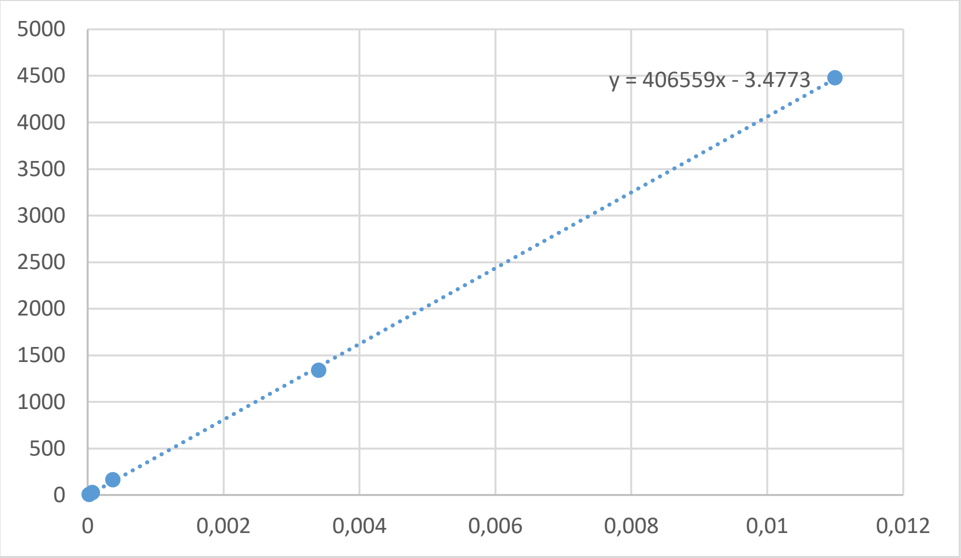
(b)

r^3 (AU^3)	T^2 (day^2)
0.000069	28.8
0.00037	166
0.011	4480
0.000022	9.692
0.44	187000
0.0034	1340

Plotting these data with the exception of planet *f*, the slope of the straight line is $4.07 \cdot 10^5$ day^2/AU^3 = $9.00 \cdot 10^{-19}$ s^2/m^3.

From Kepler's third law this is equal to $\frac{4\pi^2}{\gamma M}$,

so $M = 6.6 \cdot 10^{29}$ kg.



1.12 (a) The radius of Kepler-10b is

$$R_K = 1.4 \cdot 6.37 \cdot 10^6 = 8.9 \cdot 10^6 \,\mathrm{m}$$

$$V = \frac{4}{3}\pi \cdot R^3 =$$

$$= \frac{4}{3}\pi \cdot \left(8.9 \cdot 10^6\right)^3 = 3.0 \cdot 10^{21} \,\mathrm{m}^3$$

$$M = \rho \cdot V =$$

$$= 8.8 \cdot 10^3 \cdot 3.0 \cdot 10^{21} = 2.6 \cdot 10^{25} \,\mathrm{kg}$$

(b) The gravitational acceleration on its surface is

$$g = \frac{\gamma M}{R^2} = \frac{6.67 \cdot 10^{-11} \cdot 2.6 \cdot 10^{25}}{\left(8.9 \cdot 10^6\right)^2} = 22 \frac{\mathrm{m}}{\mathrm{s}^2},$$

more than twice that on Earth.

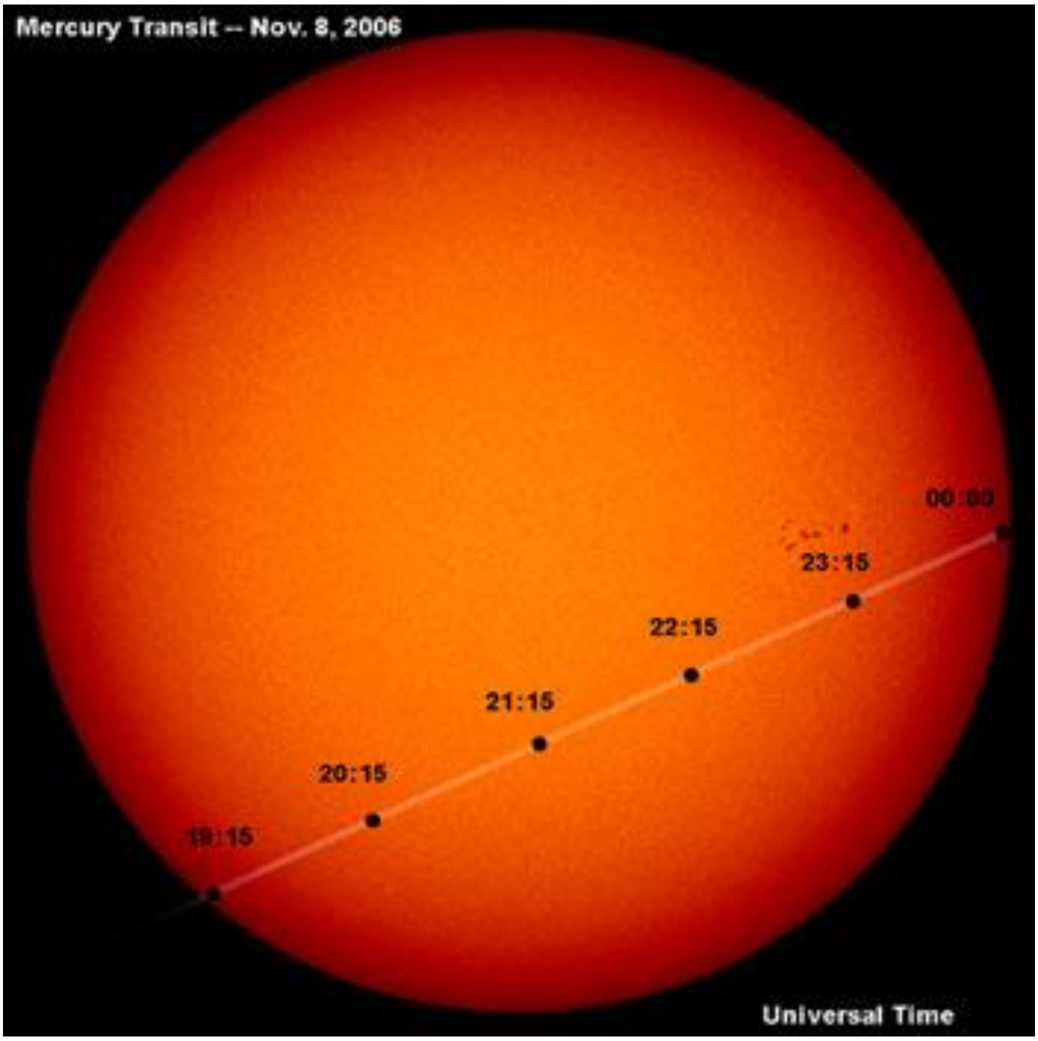
(c) On Earth: $mg = 60 \cdot 9.8 = 588\,\mathrm{N}$

On exoplanet Kepler-10b:

$$mg = 60 \cdot 22 = 1300\,\mathrm{N}$$

2. Exoplanet search methods
THE TRANSIT METHOD

2.1 The picture shows Mercury's transit in front of the Sun on 8 November 2006. It can be observed that Mercury's black disc reduces the Sun's bright area. This means that when observed from Earth, the Sun fades slightly during Mercury's transit.
(a) The Sun's radius is 696,000 km, Mercury's radius is 2,440 km. By what percent does the bright area of the Sun's disc decrease upon Mercury's transit, if we neglect the fact that Mercury is closer to us than the Sun?

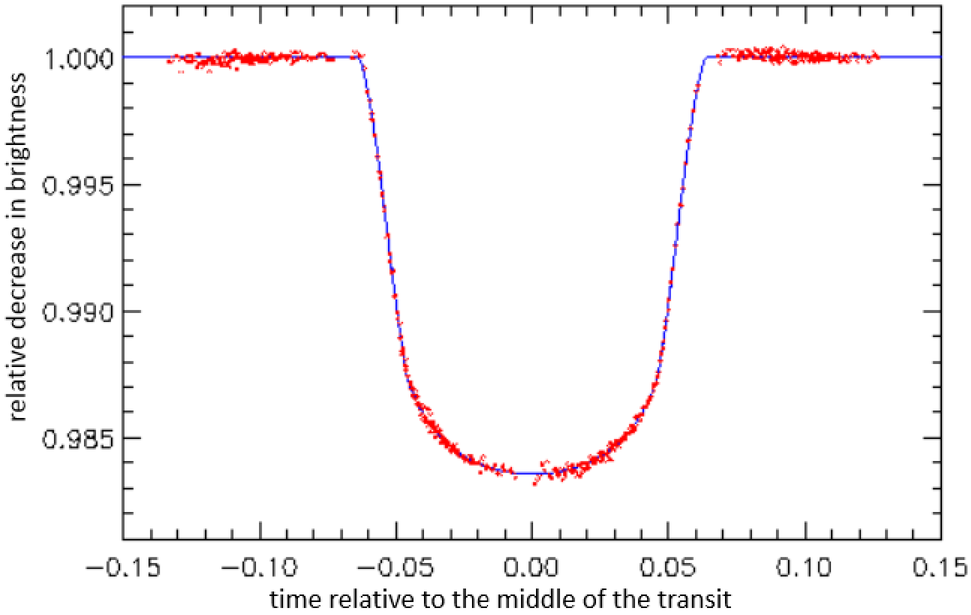


Mercury's transit in November 2006 (NASA)

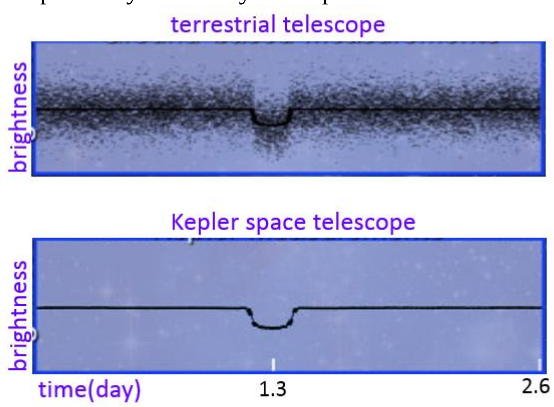
(b) How does the above result change if we also take the difference in distance into account? Calculate with a mean Mercury-Sun distance of 0.4 AU.

(c) A distant astronomer observes the Solar System. Because of the great distance he/she cannot see the Sun's disc with his instruments but when one of its planet transits in front it, its brightness still decreases. By what percent is the bright area of the Sun's disc reduced if Earth's disc or Jupiter's disc transits in front of it?

2.2 NASA's Kepler Space Telescope, which was launched in 2009, has been observing about 150,000 stars for several years. The Kepler Space Telescope measured the brightness of the stars with high accuracy to reveal the slight decrease in the brightness caused by exoplanets passing in front of them.
The graph is a light intensity curve showing the transit of a large, Jupiter-like exoplanet. The vertical axis shows the relative decrease in the brightness of the star, the horizontal axis shows the time of the transit.
In how many hours does the planet pass completely in front of the star?



Remark:
Why should the star's brightness be measured with space telescopes? The following figure shows the light intensity curve of star HAT-P-7 using the measurement data of a terrestrial telescope (top curve) and the Kepler Space Telescope (bottom curve). The inaccuracy of terrestrial data is primarily caused by atmospheric disturbances.



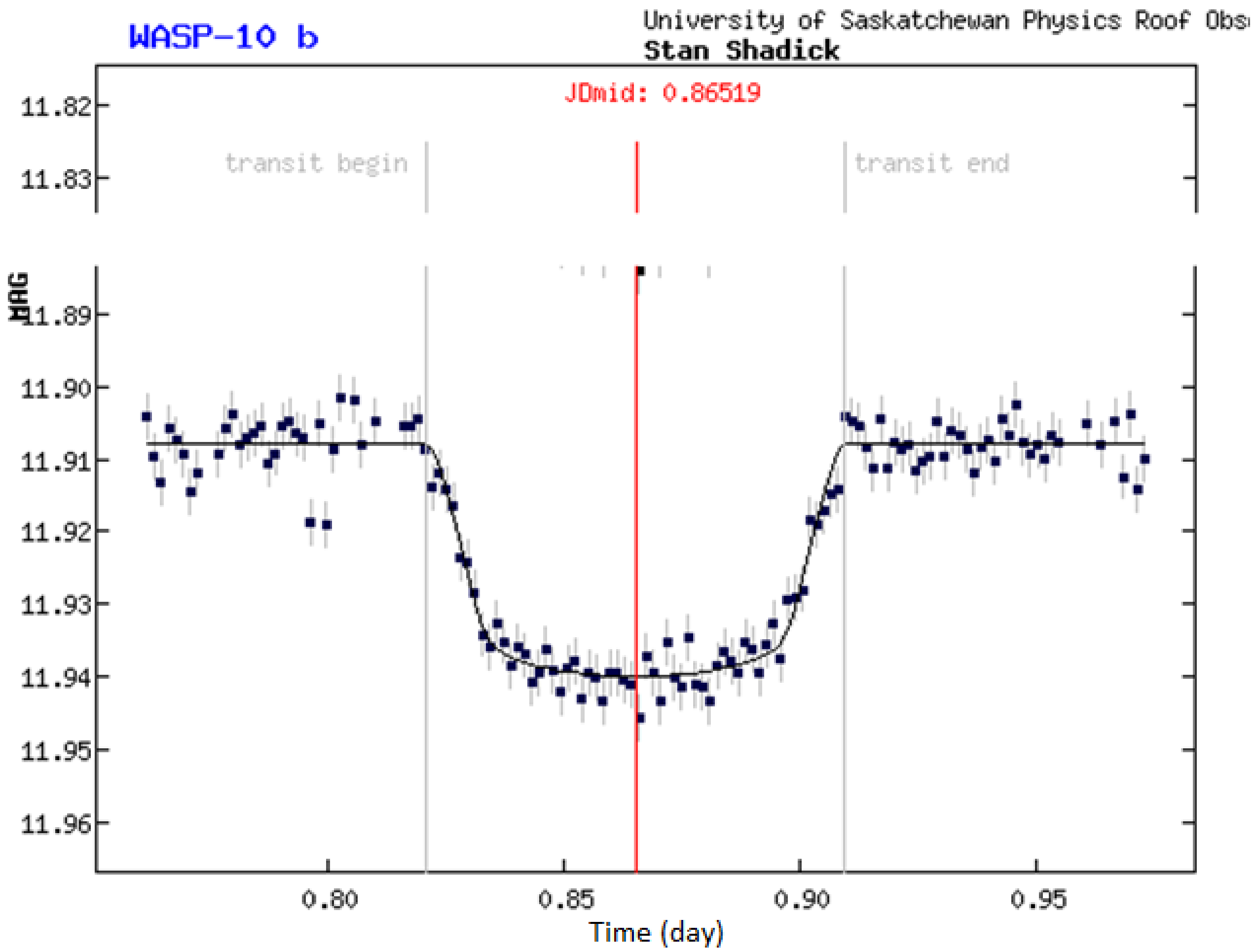
(NASA Ames Research Center)

HATNet, that is, Hungarian Automated Telescope Network, a network that consists of small, 11-cm diameter astronomical telescopes has discovered almost one hundred exoplanet systems including the one above. It was created by Gáspár Bakos and several Hungarian astronomers work in the program.

2.3 The mass of a star in the Pegasus constellation named WASP-10 by the planet searchers (unobservable to the naked eye) is 0.8 times the Sun's mass. Fortunately the planets orbital plane does not enclose a large angle with our direction of view, so we can witness transits.
The next figure shows the light intensity curve of the star: it shows the apparent brightness m of the star as the function of time. The higher the value of m, the dimmer the star. If m increases by a given value, the luminous intensity of the star decreases by the 0.3981 raised to the same value.
(a) By what percent does the luminous intensity of WASP-10 decrease during the transit of the exoplanet?

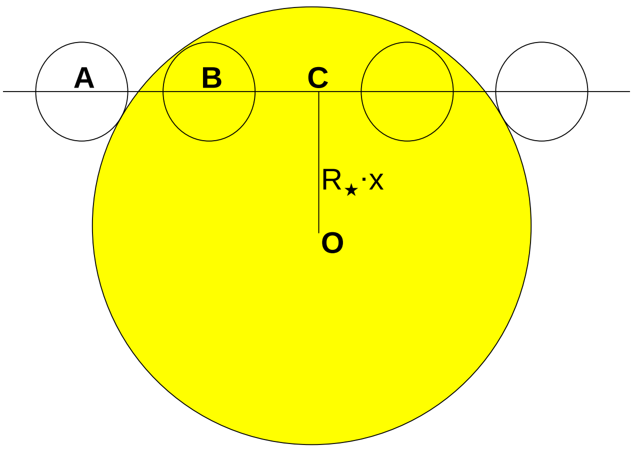
(b) Determine the ratio of the radius of the planet R to the radius of the star $R_\star$ from the given light intensity curve.

(c) Read from the diagram the ratio of the time spent by the complete planet in front of the star to the time of the transit.



http://var2.astro.cz/ETD/

(d) From the durations and the ratio of the radii determine "how far" from the centre of the star, in terms of the radius of the star, the planet passes. That is, find the value of x.

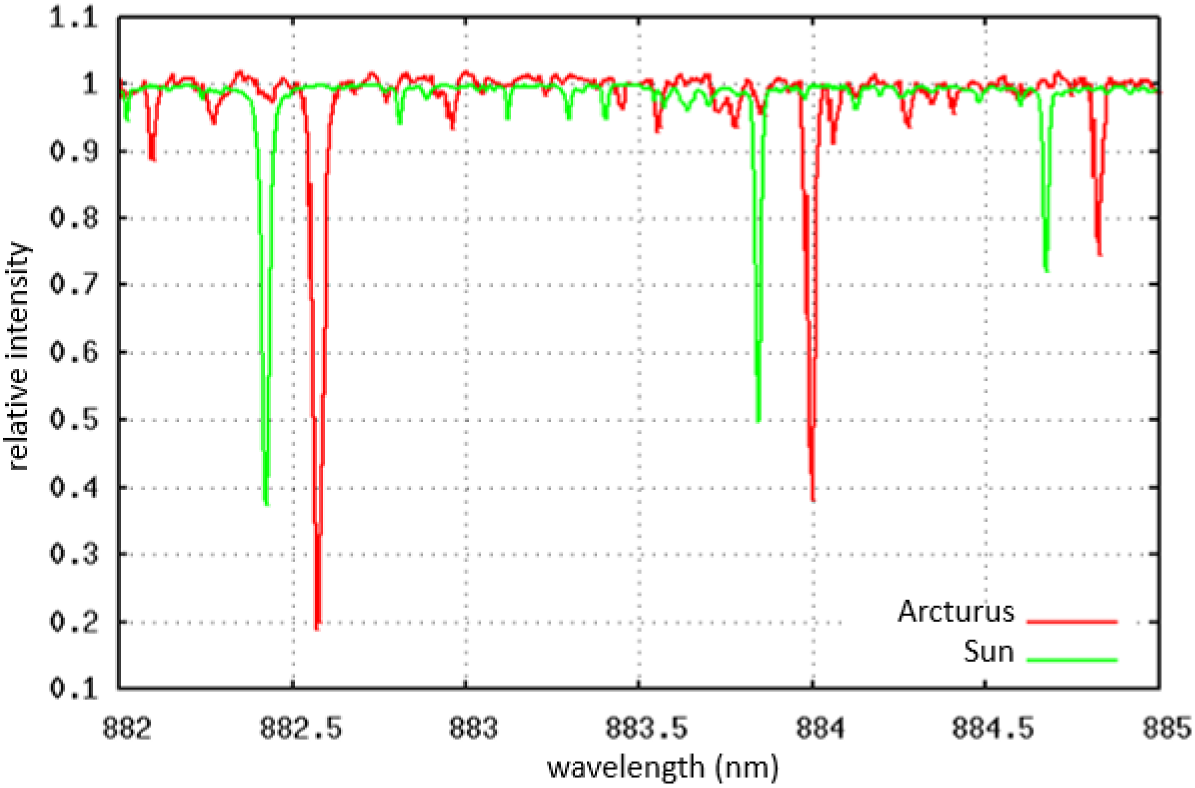


2. Exoplanet search methods
THE DOPPLER METHOD

2.4 If the distance between a source emitting light of wavelength λ and the instrument detecting it changes, then the wavelength measured by the instrument is shifted by $\Delta\lambda$ relative to value measured by the instrument if the source is stationary. The relative change in wavelength is equal to the ratio of the speed of approach or recession v and the light speed c (if v is much smaller than c):

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$$

The figure shows a small part of Arcturus' near infrared spectrum, which has characteristics very similar to the Sun's spectrum. Find Arcturus' radial speed (speed in the direction of view). Is it approaching or receding?



http://cas.sdss.org/dr7/en/proj/advanced/spectraltypes/

2.5 Estimating the order of magnitude:

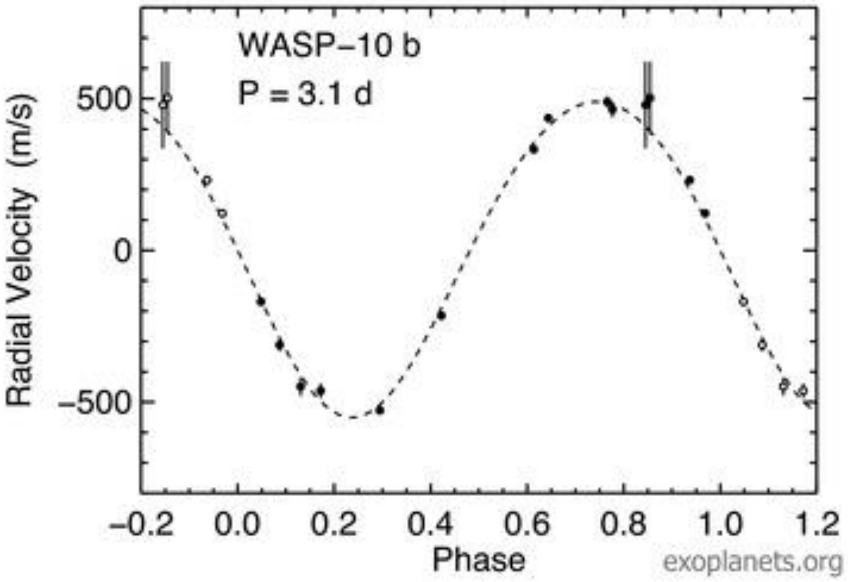
(a) Find the radius of the Sun's orbit around the centre of gravity of the Sun-Jupiter system.

(b) Find its orbital speed.

(c) A distant astronomer observes the Sun from a distance of 10 pc. He/she uses a supersensitive spectrograph that can detect a Doppler shift $\Delta\lambda/\lambda$ of one millionth. Can he/she discover Jupiter?

(d) How (in what direction) should Jupiter's mass and/or its orbital radius change so that the distant astronomer gets a chance to discover it?

2.6 Many of the planets found outside the Solar System were detected using the Doppler method: the gravitational attraction of the planet causes changes in the speed of the star. If the motion has a radial component (in the direction of observation) then the spectral lines of the star are shifted relative to the spectral lines of the stationary gas used as a reference.
The mass of star WASP-10 in the Pegasus constellation (invisible to the naked eye) is 0.8 times the Sun's mass. The following figure shows the changes in the speed of the star due to its planet WASP-10b. As shown in the figure, the period of the speed fluctuation is 3.1 days (corresponding to value 1.0 on the horizontal axis).



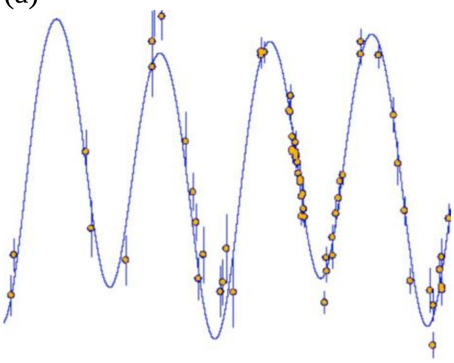
(a) Estimate the orbital radius from Kepler's Law. Compare with the Sun-Earth distance.

(b) Read the amplitude of the speed fluctuation in the diagram. What was the maximum relative Doppler shift of the star's spectrum from which this data was acquired? (And even smaller values can be measured!)

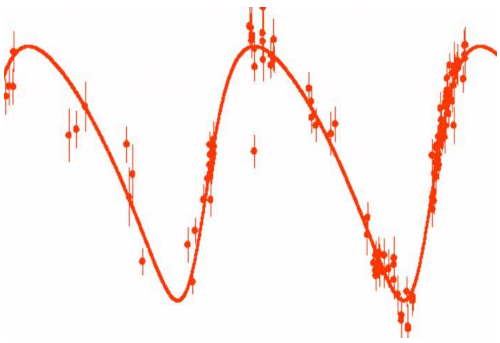
(c) From the amplitude of speed fluctuation, give a lower estimate of the planet's mass and compare it to Jupiter's mass. Why is the result only a lower estimate?

2.7 The graphs show the measured change in the radial speed of a distant star as a function of time. What can cause the deviation from the sine curve?

(a)



(b)



2.8 The following table shows the measured values of the speed of star Pegasus 51 in the Pegasus constellation as a function of time measured in days. (The inaccuracy of the speed values is only about ± 5 m/s.)

Remark:
51 Pegasi b was the first exoplanet discovered that orbits around a star similar to the Sun.

Time (day)	**Speed (m/s)**
0.62	56
0.71	67
2.60	-35
3.64	-33.5
3.82	-23
6.65	-23
7.61	-44
7.66	-34
8.61	25
8.75	41
9.60	61
10.66	-2.5
10.71	1
10.75	-5
11.69	-39
12.61	3

(Data taken from: Marcy, G. W., Butler, R. P., Williams, E., Bildsten. L., Graham, J. R., Ghez, A. M., & Jernigan, J. G. 1997, Astrophysical Journal, 481, 926)

(a) Plot the speed of the star as a function of time and determine the orbital period of the planet.

(b) Find the length of the semi-major axis of the orbit in AU assuming that the mass of Pegasus 51 is equal to the mass of the Sun.

(c) Give an estimate of the mass of the planet. Compare it to the mass of Earth and Jupiter.

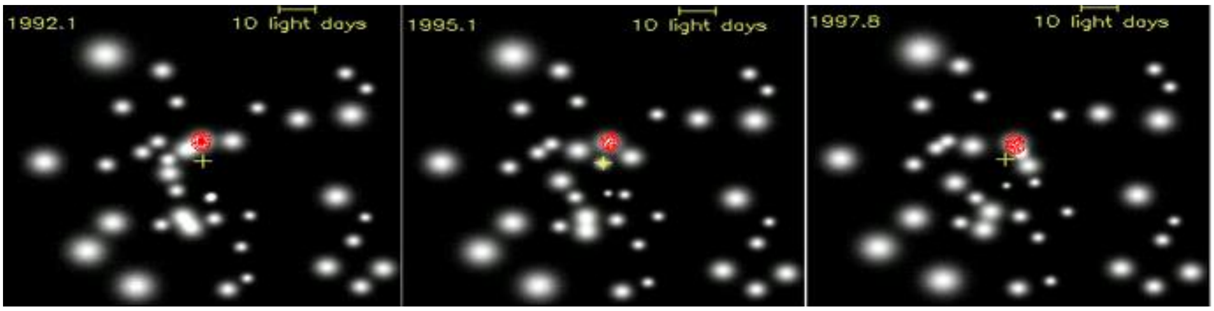
2. Exoplanet search methods
ASTROMETRIC METHOD

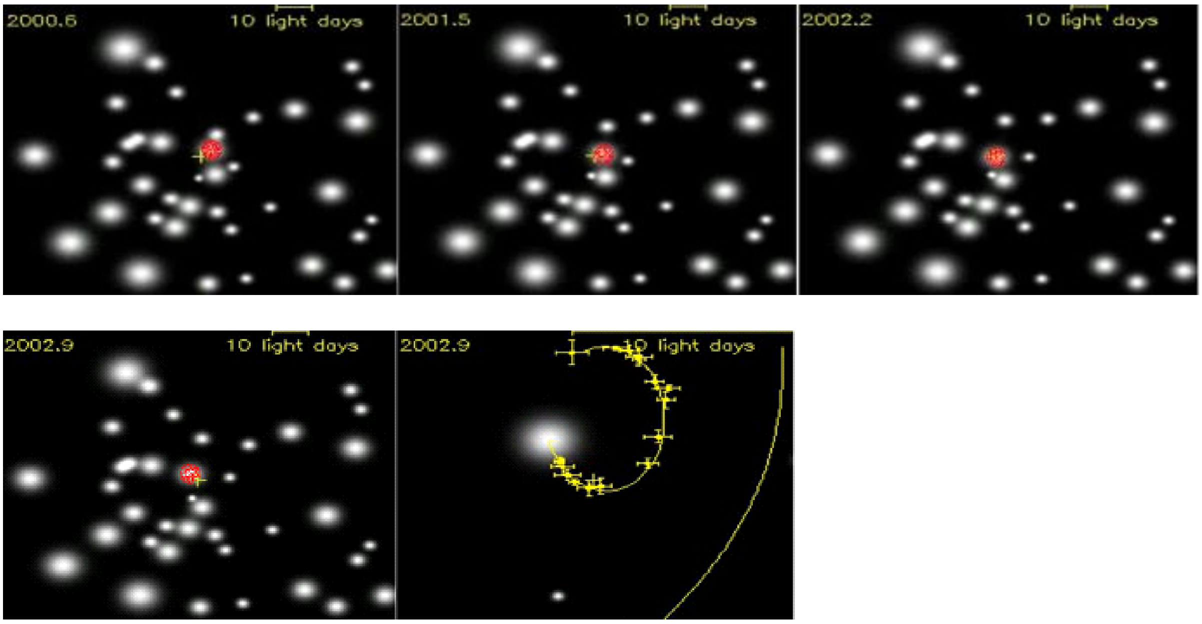
2.9 Somewhere in the Galaxy, an astronomer looking for planets observes the Sun. Find the distance of the most distant astronomer who can have the a to conclude Jupiter's presence from the Sun's wobbling if the smallest wobbling that can be detected with his/her instruments is in the magnitude 10^{-4} arc seconds? (The effect of the other planets is neglected, as the vast majority of the mass of the Solar System outside the Sun is concentrated in Jupiter. You can also assume that its orbit is a circle.)

2.10 (*Based on a Student Olympiad preparatory course problem*)
The accuracy required to detect the wobbling of stars has not been available until now, so for the time being, we know only a few exoplanets discovered using astrometry.
The Gaia Astrometry Satellite was launched by the European Space Agency (ESA) in December 2013. The space telescope orbits around the Sun on Earth's orbit, its aim is to measure the exact data of the stars in the Milky Way. Within the frame of the Gaia mission, hundreds of thousands of exoplanet discoveries are expected in the coming years. In addition, exoplanets within a distance of about 1,600 light years (long-period gas giants) will be discovered with this method, which could not be found by other methods.
(a) The space telescope is able to detect a change of 2×10^{-5} arc-seconds in the direction of the stars. If such a change can be observed in the direction of a star at a distance of 1,000 light-years, what is the orbital radius in AU of the star around the centre of gravity of the system consisting of the star and its planet?

(b) The size of Gaia's main mirror is 1.5 m × 0.5 m. The light falling on the mirror is collected by the 1-gigapixel detector whose size is 0.5 m × 1 m. Estimate the number of photons arriving from the star on the detector in one second.

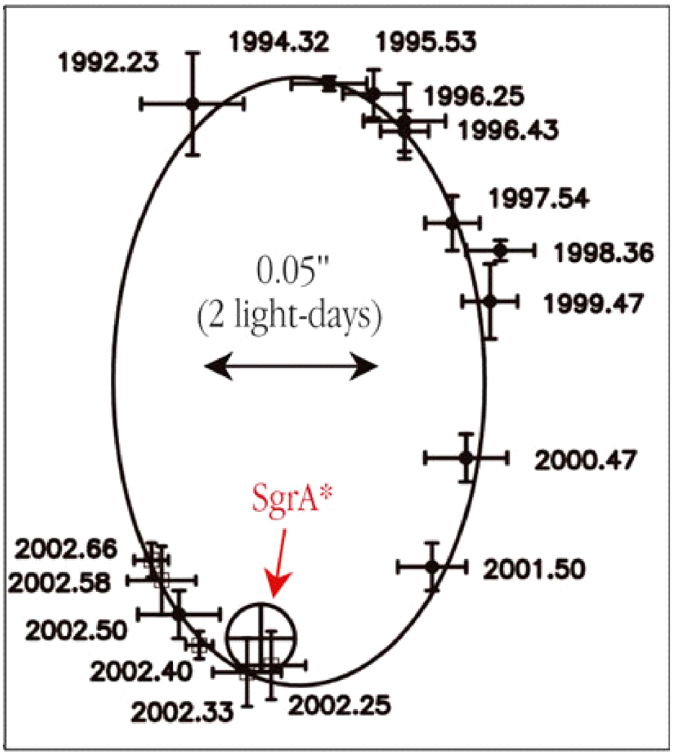
2.11 The astrometric method, which is becoming more and more applicable in the exploration of exoplanets, is easier to study when we deduce from the star's own movement a partner not of smaller, but of larger mass.
The first seven images of the following series of images show the same region of the sky (the small cross marks the middle) at several times during the ten years from 1992 to 2002. The star marked with red is visibly changing its position. Based on the distance of the star, the change of position is about 10 light-days. The eighth image shows the identified positions of the star during the ten years in magnification (with the estimated error of measurement). The points clearly form an ellipse, the star is orbiting around something that does not appear in the images.
(a) Find the displacement compared to the size of the Solar System: compare the displacement of the star with the diameter of Neptune's orbit.





http://astronomie-smartsmur.over-blog.com

(b) The plane of the elliptical orbit is not necessarily perpendicular to our direction of view, so because of the perspective the direction of the major axis is not necessarily what we see as the greatest extension of the ellipse. However, based on Kepler's law of areas, it is possible to determine the centre of attraction around which the star is orbiting. This object received the name Sagittarius A * (Sgr A *) from the Latin name of the constellation, Sagittarius, its position is marked by the cross in the figure below. Assuming that the length of the major axis is 10 light-days, give an estimate of the mass of Sagittarius A* based on the figure.



http://astronomie-smartsmur.over-blog.com

(c) Give an upper estimate of the radius of Sagittarius A* based on the figure.

Solutions 2.

2.1 (a) The percent of the dark area is:

$$\frac{R_{\mathrm{M}}^2}{R_{\mathrm{S}}^2}=\frac{2440^2}{696000^2}=1.2\cdot 10^{-5}=0.0012\%$$

(b) If the three celestial bodies are in the same line, then Mercury's distance is 0.6 times the Sun's distance.

If Mercury was at the same distance as the Sun, it would cover $\dfrac{1}{0{,}6^2}\approx 3$ times larger area, so the percent of the dark area is approximately three times the calculated value.

Remark:
If Mercury's disc is replaced by the Moon's disc, the Moon's disc covers the Sun completely, causing a total solar eclipse. The other extreme is when the star and its planet are so far from the observer that the difference in the distance is truly negligible: this is the situation when the transit of an exoplanet is observed in front of its star.

(c) Earth:

$$\frac{R_{\mathrm{E}}^2}{R_{\mathrm{S}}^2}=\frac{6370^2}{696000^2}=8.4\cdot 10^{-5}=0.0084\%$$

Jupiter: $\dfrac{R_{\mathrm{J}}^2}{R_{\mathrm{S}}^2}=\dfrac{71500^2}{696000^2}=0.0106=1.1\%$

2.2 Approximately 0.12 hours.

2.3 (a) The approximate change in magnitude as read from the figure is Δm = 11.941 – 11.912 = 0.039.

The relative change in magnitude is
$1 - 0.3981^{0.039} = 1 - 0.965 = 3.5\%$.

(b) Expressed from the ratio of the covered area:

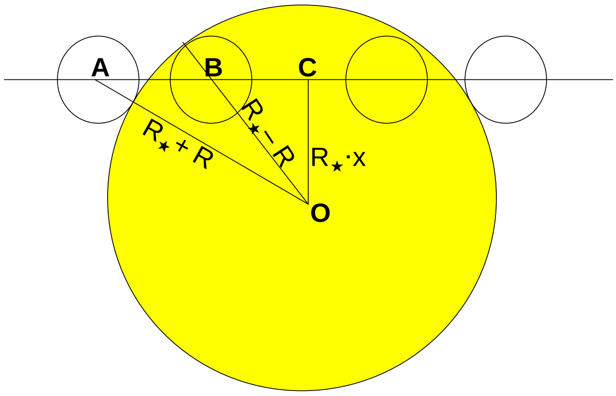
$$\frac{R_*^{\,2}-R^2}{R_*^{\,2}}=1-\left(\frac{R}{R_*}\right)^2=0.965\,,$$

$$\text{so}\quad \frac{R}{R_*}=0.19\,.$$

(c) The duration of the horizontal part, t_2 should be divided by the total duration of the decrease, t_1. This can be done by measuring these values in the figure with a ruler. It is a rough estimate, because it is difficult to decide where the curve breaks.

$$\frac{t_2}{t_1}=\frac{55}{88}\approx 0.63\,.$$

(d) Based on the figure, the distances covered in times t_1 and t_2 are 2AC and 2BC, expressing their squares from Pythagoras' theorem and dividing them by each other:



$$\left(\frac{t_2}{t_1}\right)^2=\frac{(R_*-R)^2-R_*^2x^2}{(R_*+R)^2-R_*^2x^2}=$$

$$=\frac{\left(1-\dfrac{R}{R_*}\right)^2-x^2}{\left(1+\dfrac{R}{R_*}\right)^2-x^2}$$

$$0.63^2=\frac{(1-0.19)^2-x^2}{(1+0.19)^2-x^2}$$

$$0.562-0.63x^2=0.656-x^2$$

$$x=0.48\approx 0.5$$

2.4 The Sun's distance does not change. If on the horizontal axis of the graph the distance between the marks is 2.5 cm, then the shift in the spectral lines is 0.8 cm, which corresponds to $\Delta\lambda$ = +0.16 nm on the scale.

(There is no noticeable difference between the shift of the three lines, so let us consider the one in the middle.)

The speed

$$v=\frac{\Delta\lambda}{\lambda}\cdot c=\frac{0.16}{883.84}\cdot 3.0\cdot 10^8=54\ \mathrm{km/s}$$

In Arcturus' spectrum the wavelengths are longer, so it is moving away from us.

2.5 (a) The mass ratio of the Sun and Jupiter is

$$\frac{2.0\cdot 10^{30}}{1.9\cdot 10^{27}}\approx 1000\,,$$

so the centre of mass is approximately one thousandth of Jupiter's orbital radius from the Sun's centre.

The requested radius is therefore
$0.001r = 0.001 \cdot 7.8 \cdot 10^{11}$ m $\approx 8 \cdot 10^{8}$ m.

(b) Jupiter's orbital period is approximately 12 years, so

$$v = \frac{2r\pi}{T} = \frac{2\pi \cdot 8 \cdot 10^{8}}{1.9 \cdot 10^{8}} = 13 \text{ m/s}$$

(c) The Doppler shift due to Jupiter is only

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{13}{3 \cdot 10^{8}} \approx 4 \cdot 10^{-8} << 1 \cdot 10^{-6},$$

which cannot be detected.

2.6 (a) The period of the speed fluctuation is the orbital period of the planet, so

$$T = 3.1 \cdot 24 \cdot 3600 = 2.7 \cdot 10^{5} \text{s}.$$

If $M >> m$,

$$r = \left(\frac{\gamma M T^2}{4\pi^2}\right)^{1/3} =$$

$$= \left(\frac{6.67 \cdot 10^{-11} \cdot 0.8 \cdot 2.0 \cdot 10^{30} \cdot (2.7 \cdot 10^{5})^2}{4\pi^2}\right)^{1/3}$$

$$r = 5.8 \cdot 10^{10} \text{m} \approx 0.04 \text{ AU}.$$

(b) The amplitude of speed fluctuation is approximately
$v_{max} = 510$ m/s.

$$\frac{\Delta\lambda}{\lambda} = \frac{v_{max}}{c} = \frac{510}{3.0 \cdot 10^{8}} = 1.7 \cdot 10^{-6}.$$

(c) The orbital radius of the star around the centre of mass of the system is

$$r' = \frac{m}{M+m} r \approx \frac{m}{M} r$$

$$v_{max} = \frac{2r'\pi}{T} = \frac{2\pi m}{MT} \cdot r,$$

so from exercise part (a)

$$v_{max} = \frac{2\pi m}{MT} \cdot \left(\frac{\gamma M T^2}{4\pi^2}\right)^{1/3}$$

$$m = \left(\frac{M^2 T}{2\pi\gamma}\right)^{1/3} \cdot v_{max}$$

$$m = \left(\frac{(1.6 \cdot 10^{30})^2 \cdot 2.7 \cdot 10^{5}}{2\pi \cdot 6.67 \cdot 10^{-11}}\right)^{1/3} \cdot 510$$

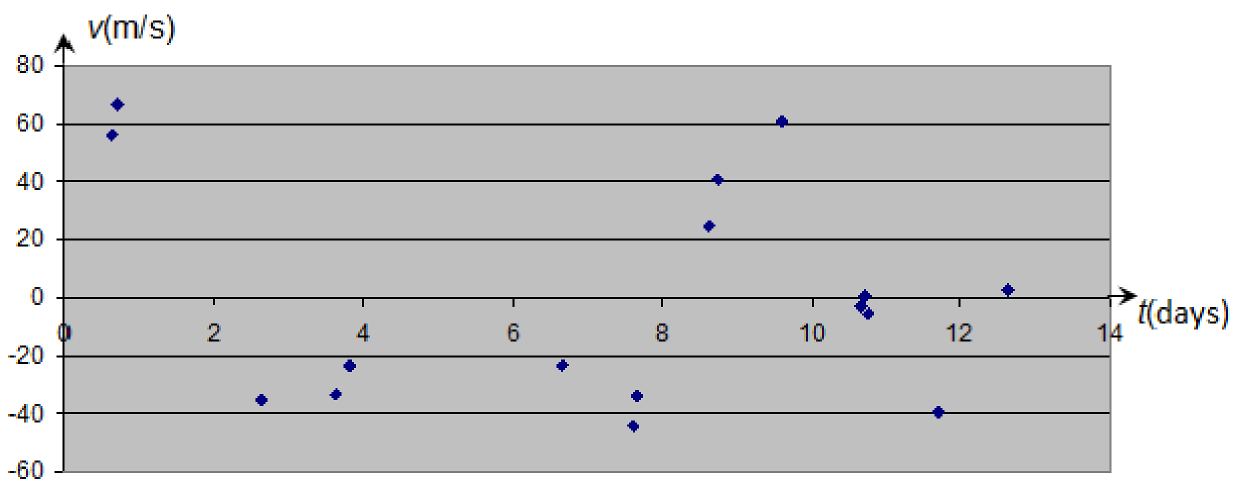
$$m = 6.1 \cdot 10^{27} \text{kg} \approx 3.2\, m_{\text{Jup}}$$

It is only a lower estimate because we do not know the plane of the orbit of the orbiting planet. The maximum radial speed is the same as the actual speed only if the observer is in the plane of the orbit. If the plane encloses an angle with the direction of view, the detected speed is less than the actual speed.

2.7

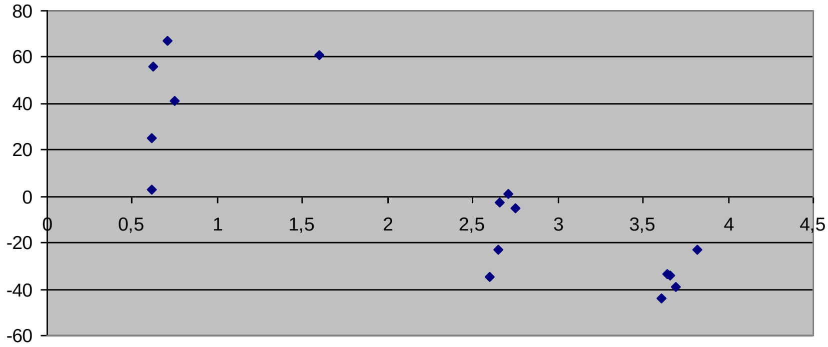
(a) Several planets orbit around the star.
(b) The orbit of the planet is highly eccentric.

2.8 (a) The drawn points should be on a sine curve, but the graph does not show a noticeable periodicity. There is a maximum somewhere between 1 and 2 days, there is no data at the next maximum, then there is a peak somewhere between 9 and 10 days. Based on this we can try a 4-day period: consider day mod 4.

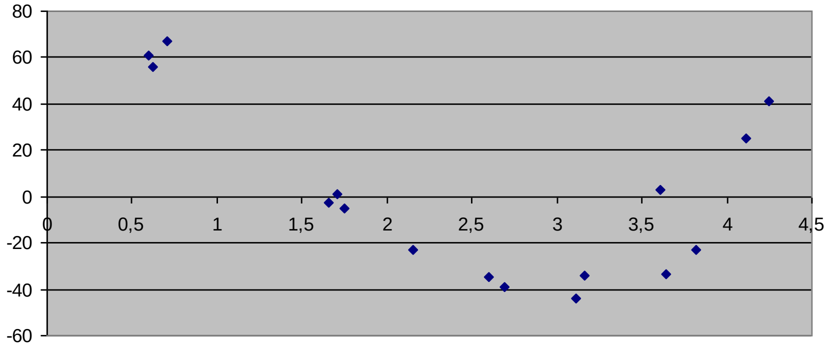


Time (day mod 4)	Speed (m/s)
0.62	56
0.71	67
2.60	-35
3.64	-33.5
3.82	-23
2.65	-23
3.61	-44
3.66	-34
0.61	25
0.75	41
1.60	61
2.66	-2.5
2.71	1
2.75	-5
3.69	-39
0.61	3

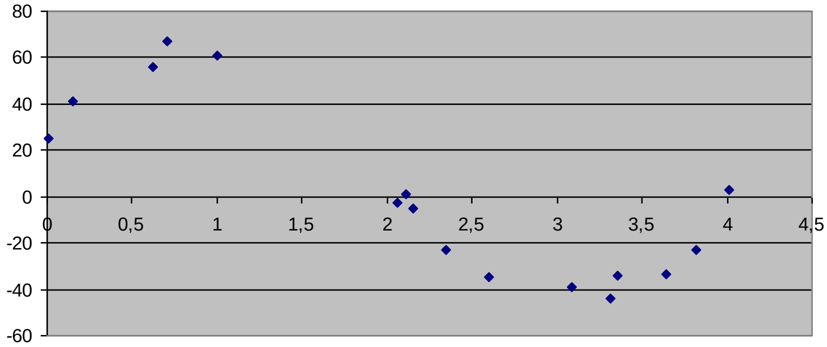
Period of 4 days:



4.5 days



4.25 days



When trying slightly different periods, we find that 3.5 days give worse, 4.5 days give slightly better results. This is closer to a curve, but not a sinusoidal curve (we have found the period if the graph shows a single period of a sine function).

Refining further, we can find that the period is about 4.25 days.

(b) For a star whose mass is equal to the Sun's mass, the square of the period in years is equal to the cube of the semi-major axis in astronomical units.

$$T = 4.25 \text{ days} = 0.0116 \text{ years}$$

$$a = 0.0116^{2/3} = 0.0514 \approx 0.05 \text{ AU}$$

So the planet orbits around its star at only one twentieth of Earth's orbital radius. (Approximately one eighth of Mercury's orbital radius.)

(c) For the estimate, suppose that the orbit of the planet is circular. Then the speed of the planet is

$$v_P = \frac{2\pi \cdot a}{T} = \frac{2\pi \cdot 0.0514 \cdot 1.5 \cdot 10^{11}}{4.25 \cdot 24 \cdot 3600} =$$
$$= 130000 \text{m/s}.$$

Suppose, furthermore, that we see the "edge" of the plane of the orbit, that is, the maximum speed read from the above graph is the speed of the star. The maximum value in the graph is approximately +65 m/s, the minimum is approximately -45 m/s, so the amplitude of the speed fluctuation is about 55 m/s.

The speeds are inversely proportional to mass, so the mass of the planet is

$$m_P = m_{Sun} \cdot \frac{v_{star}}{v_P} = 2 \cdot 10^{30} \cdot \frac{55}{130000} =$$
$$= 8 \cdot 10^{26} \text{kg}.$$

That is approximately 140 times Earth's mass and a bit less than half of Jupiter's mass.

2.9 The Sun's mass is $2.0\cdot10^{30}$ kg, Jupiter's mass is $1.9\cdot10^{27}$ kg, their distance is $7.8\cdot10^{11}$ m. The Sun orbits around the centre of mass of the system on an orbit whose radius is

$$\frac{m}{M+m}r \approx \frac{m}{M}r = \frac{1.9\cdot10^{27}}{2.0\cdot10^{30}}\cdot 7.8\cdot10^{11}\,\mathrm{m} =$$
$$= 7.4\cdot10^{8}\,\mathrm{m},$$

so its diameter is $1.5\cdot10^9$ m. The given angle in radians is

$$10^{-4}\cdot\frac{2\pi}{360\cdot60^2} = 4.8\cdot10^{-10}\,\mathrm{rad},$$

so the requested distance is in the magnitude

$$\frac{1.5\cdot10^{9}}{4.8\cdot10^{-10}} = 3.1\cdot10^{18}\,\mathrm{m} \approx 100\ \mathrm{pc}.$$

2.10 (a) The orbital radius is seen in an angle of $1\cdot10^{-5}$ angular seconds from a distance of 1000 light-years.

1000 light-year = $9.5\cdot10^{18}$ m,

$1\cdot10^{-5}$'' = $4.8\cdot10^{-11}$ rad, so

$$r = 9.5\cdot10^{18}\cdot4.8\cdot10^{-11} = 4.6\cdot10^{8}\,\mathrm{m}.$$

Remark:
This result is already in the order of magnitude of the orbital radius of the Sun around the centre of mass of the Sun-Jupiter system. (See Exercise 2.9).

(b) Consider the distant star to be similar to the Sun, then its luminosity is $4\cdot10^{26}$ W, its intensity maximum is at the wavelength 500 nm, let us use the latter data to find the energy of a single photon:

$$E = \frac{hc}{\lambda} = \frac{6.6\cdot10^{-34}\cdot3\cdot10^{8}}{5\cdot10^{-7}} = 4\cdot10^{-19}\,\mathrm{J}$$

The number of photons emitted per second is

$$\frac{L}{E} = 1\cdot10^{45}$$

This number of photons is distributed on a sphere whose radius is $9.5\cdot10^{18}$ m, the detector only detects the fraction that passes through the 0.75-m^2 mirror of the telescope:

$$1\cdot10^{45}\cdot\frac{0.75}{4\pi\cdot(9.5\cdot10^{18})^2} \approx 660000\ \text{photons}.$$

2.11 (a) Neptune's orbital radius is 30 AU. Its diameter is 60 AU = 0.35 light-days, the displacement of the star is approximately 30 times this value.

(b) The length of the semi-major axis is

$a = 5\cdot24\cdot3600\cdot3\cdot10^8 = 1.3\cdot10^{14}$ m.

The time belonging to the position at one of the endpoints of the major axis is 2002.33 years. Before that, the time belonging to the other endpoint is approximately

$$\frac{1994.32+1995.53}{2} = 1994.9\ \text{years}$$

The difference is 7.4 years, that is, the orbital period is about 15 years = $4.7\cdot10^8$ s.

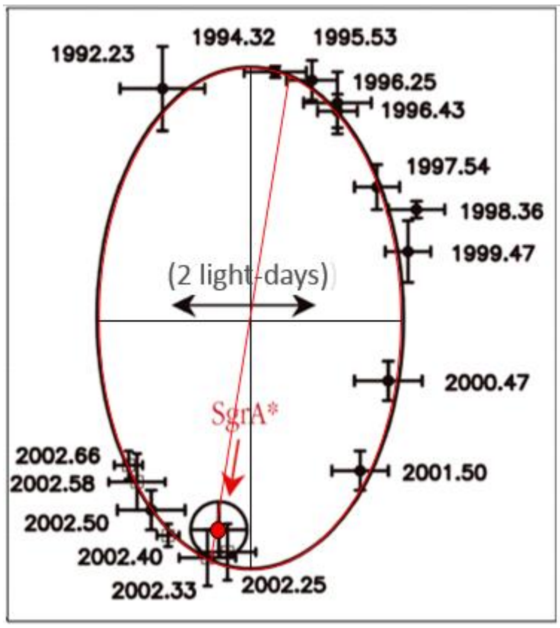
$$\frac{a^3}{T^2} = \frac{\gamma M}{4\pi^2}$$

$$M = \frac{4\pi^2a^3}{\gamma T^2} = \frac{4\pi^2\cdot(1.3\cdot10^{14})^3}{6.7\cdot10^{-11}\cdot(4.7\cdot10^{8})^2} \approx$$
$$\approx 6\cdot10^{36}\,\mathrm{kg}$$

This is three million times the Sun's mass.

(c) The star orbiting around it approaches object Sagittarius A* to approximately 0.5 light-days = $1.3\cdot10^{13}$ m, that is, less than 90 AU. So it is definitely smaller than that.

Remark:
An object of such a large mass compressed in such a small space cannot be else than a black hole.



3. Cosmic research and its tools

NEAR-EARTH ASTEROIDS AND OTHER TINY CELESTIAL BODIES

3.1 On 14 February 2013, a 10,000-tons meteor struck into the atmosphere at a speed of 18 km/s. It exploded above the city of Chelyabinsk in Russia; the detonation caused considerable damage and about one thousand people were injured due to glass shatter from the windows broken as a result of the shock wave. (Many witnesses made several videos about the event)
(a) How many kilotons of TNT is the energy of the Chelyabinsk meteor equivalent to, if 1 ton of TNT provides $4.2 \cdot 10^9$ J?
(b) According to statistics, on average two objects larger than 4 metres arrive on Earth's entire surface per year. 72% of Earth's surface is covered by oceans and only 3% of the land is inhabited. In how many years can we expect to hear about the impact of such a large meteor in the news?

(c) Fireballs are meteors that create a bright strip behind them in the sky that is visible from an area of approximately 100 km^2. If the event is accompanied by a sonic boom as well, it is called a bolide. The mass of a bolide is at least half a kg. It is estimated that about 50,000 pieces of such objects fall on the entire 500 million km^2 surface of Earth. Approximately how many bolides can you see in your life with your own eyes?


Right: The super-bolide of Chelyabinsk http://spacemath.gsfc.nasa.gov, Left: Lake Manicouagan (Google Earth)

3.2 The following empirical formula can be used to calculate the diameter D (in km) of a crater formed by the impact of a meteorite with energy E (in joule):

$$D = 1.96 \cdot 10^{-5} \cdot E^{0.294}$$

(a) Find the size of the crater created by a meteorite weighing $5 \cdot 10^9$ kg striking at a speed of 20 km/s.

(b) Manicouagan Lake, located in Québec, Canada (5km in diameter), was formed in the middle of a 100-km-diameter meteor crater created 214 million years ago. How many kilotons of TNT equivalent energy was released in the impact if 1 kiloton of TNT is equivalent to $4.2 \cdot 10^{12}$ J?

3.3 The brightness of an asteroid or comet seen from Earth depends on many factors. Some factors are its size, its albedo, its distance from the Sun and the time of observation, since it is important whether it is seen from the fully illuminated side (like a full moon) or a partially illuminated side.
Taking into account the many variables, the following empirical formula was found to be useful in any part of the solar system. (It is assumed that the albedo of the asteroid is like that of moon rocks.)

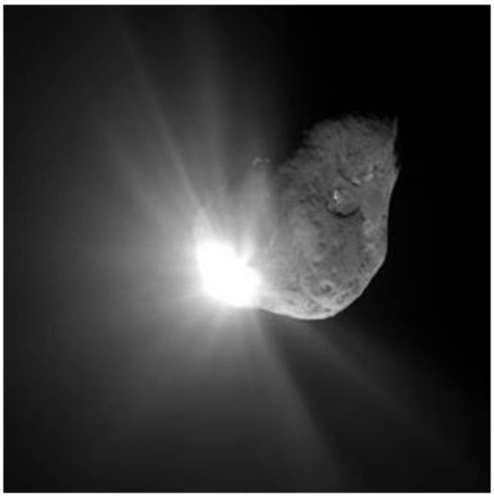
$$R = 0.011 \cdot d \cdot 10^{-m/5}$$

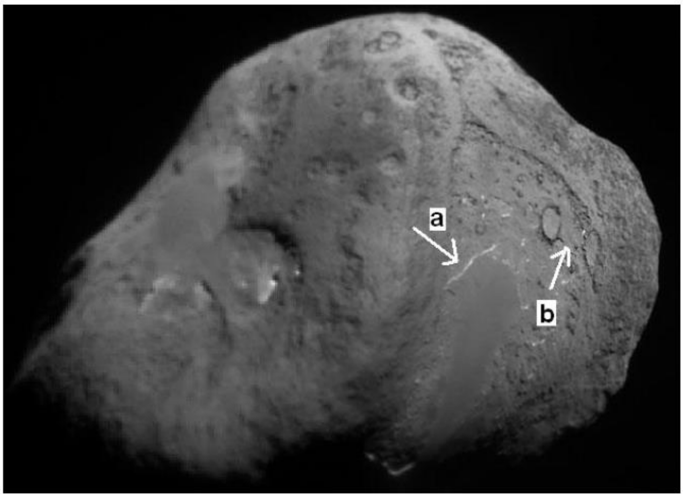
where R is the size of the asteroid in metres, d is its distance from Earth in kilometres and m is the apparent brightness of the asteroid seen from Earth. The smaller the value of m, the brighter the object. In the case of objects just visible to the naked eye, $m \approx 6$.
(a) An asteroid will get closest to Earth in 2027. Then its distance will be 37,000 km. State a numerical formula for the relationship between its size R and its apparent brightness m.

(b) Will the asteroid be visible if its size is between 200 m and 1000 m?

3.4 On 4 July 2005 the Deep Impact space probe approached the core of comet Tempel 1 to 500 kilometres. The Impactor projectile forming part of the space probe, finally struck into the comet. The first image shows the flash of light and the outflow of gas upon impact.
(a) The width of the second image (compiled from the images taken by Impactor) is 8.0 km. Find the approximate size of the core of the comet in km. Find the size of the craters that are on the right of the picture. Which are the smallest, barely detectable details of the image?

(b) The Impactor projectile hit the comet in the position indicated by the arrow **b**. What inaccuracy in its path would have prevented the encounter?





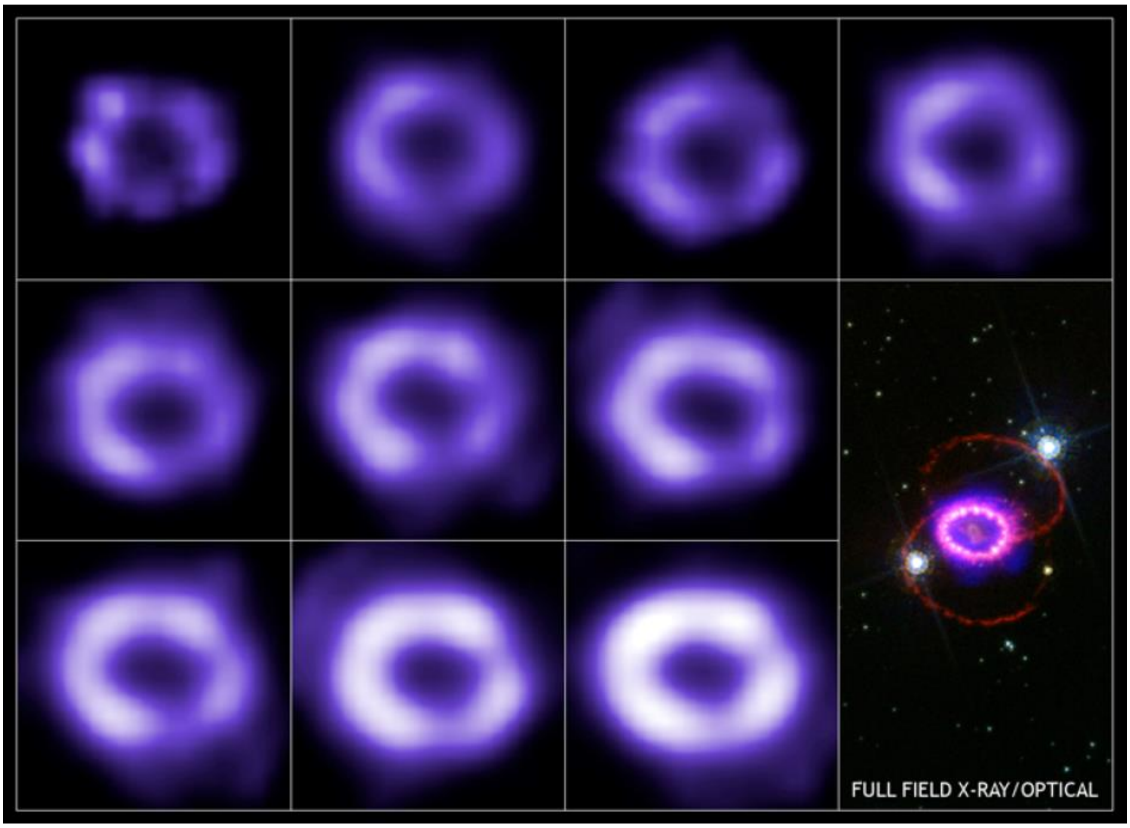
http://spacemath.gsfc.nasa.gov

3.5 (a) The core of comet Tempel 1 has an average density of 400 kg/m^3, and can be approximated with a sphere of radius 3 km. Find its mass.
(b) A crater was formed at the place of impact of the 362-kg probe that was moving at a speed of 10.3 km/s, and about 10,000 tons of material was ejected. If the Impactor hit the comet core perpendicularly to its path, what perpendicular velocity component was gained by the comet core?

(c) Imagine that this comet moves towards Earth and is calculated to strike it 50 years later. A 10 megaton TNT equivalent nuclear bomb is launched to explode when it hits the comet. When the spacecraft carrying the bomb arrives, there are still 20 years remaining from the 50 years. The explosion would transfer as much impulse to the comet as if the Impactor that hit it at a speed of 10.3 km/s had a mass of $7.5 \cdot 10^8$ kg.
Assuming that the core of the comet does not crumble into dust, but remains together, by how much is it diverted from its track in 20 years? Can collision with Earth be prevented this way?

3. Cosmic research and its tools
INTERPRETATION AND APPLICATION OF DATA PROVIDED BY SPACECRAFTS

3.6 In March 1987, a supernova explosion was observed in the nearby galaxy of the Large Magellanic Cloud, which is 160,000 light years from the Milky Way. Judging from the location of the explosion, the blue super-giant Sanduleak-69º 202a (shortly SK-69), whose mass is 20 times that of the Sun, became a supernova. The following sequence of images shows an expanding gas cloud at a temperature of about one million degrees, these were taken by the Chandra X-ray telescope between January 2000 (top left image) and January 2005 (bottom right image).
Find the average speed of the expansion if the width of each image is 1.9 light years.



http://spacemath.gsfc.nasa.gov

3.7 In 2017, the 40th anniversary of the Voyager-1 spacecraft, the man-made spacecraft that has gone furthest, is celebrated. Voyager-1 started its journey in 1977, it is moving away at a speed of 17 km/s and its communication system still works.
(a) How far is it now?

(b) How long will it take for it to cover the distance of the closest star?

(c) Jupiter's moon Io, whose diameter is 3,630 km, is characterized by intense volcanic activity. The image taken by Voyager-1 shows the eruption of the Prometheus volcano. (Volcanoes are traditionally named after the gods in the myths of different peoples.)
Based on the picture, give an estimate of how high the material thrown up by Prometheus rises.

(d) Io's density is $3.55 \cdot 10^3$ kg/m^3 (close to the density of our Moon, $3.34 \cdot 10^3$ kg/m^3). At what speed does the material thrown up leave the volcano?

3.8 The two photos shown below were taken by the SOHO satellite that orbits on Earth's orbit at the times when it was nearest to (perihelion) and farthest from the Sun (aphelion).
(a) When is it closest to and farthest from the Sun?

(b) Use the photos to determine the percent by which the two distances deviate from the average.

(c) If the average distance from the Sun is 149,600,000 km, how much closer is Earth at perihelion than at aphelion?

http://spacemath.gsfc.nasa.gov

3.9 (a) The luminosity of the Sun is $L = 3.9 \cdot 10^{26}$ W. Pluto is 31 AU from the Sun at perihelion and 49 AU at aphelion. Pluto rotates around its axis. Find the temporal average of radiation intensity in W/m^2 reaching Pluto's atmosphere at perihelion and aphelion.

(b) Before the New Horizons space probe passed near Pluto in 2015, little was known about Pluto's atmosphere. According to the data of the space probe, Pluto's atmosphere consists of methane and reflects 60% of light falling on it (i.e. Pluto's albedo is 0.6) and absorbs the rest. From this data we can estimate the temperature of the atmosphere.

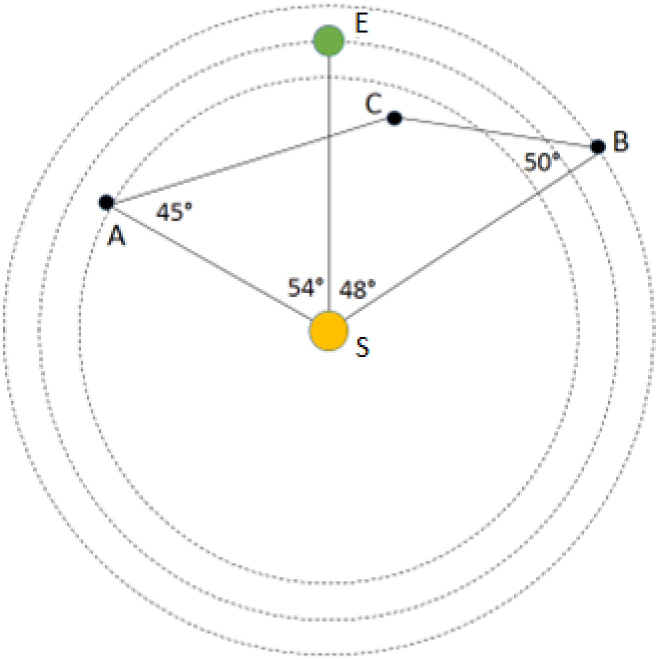
As the atmosphere is in a thermal equilibrium, the same amount of energy is emitted into space as absorbed. Find the absolute temperature T of the atmosphere at perihelion and aphelion if the intensity of the emission is $\sigma \cdot T^4$, and the value of constant σ is $5.67 \cdot 10^{-8}$ W/(m^2K^4).

3. Cosmic research and its tools

THE GEOMETRY OF MODERN SATELLITES

3.10 The STEREO satellite pair is orbiting around the Sun. One is slightly inside Earth's orbit, the other is slightly outside. Thus, they can observe the same phenomenon simultaneously from two points at a large distance, from different angles, so they can form a 3-dimensional image of solar flares, storms, and other events occurring on or near the surface of the Sun. They can also be used to study the plasma clouds moving from the Sun towards Earth. Their distance, speed, shape, etc. can be determined.

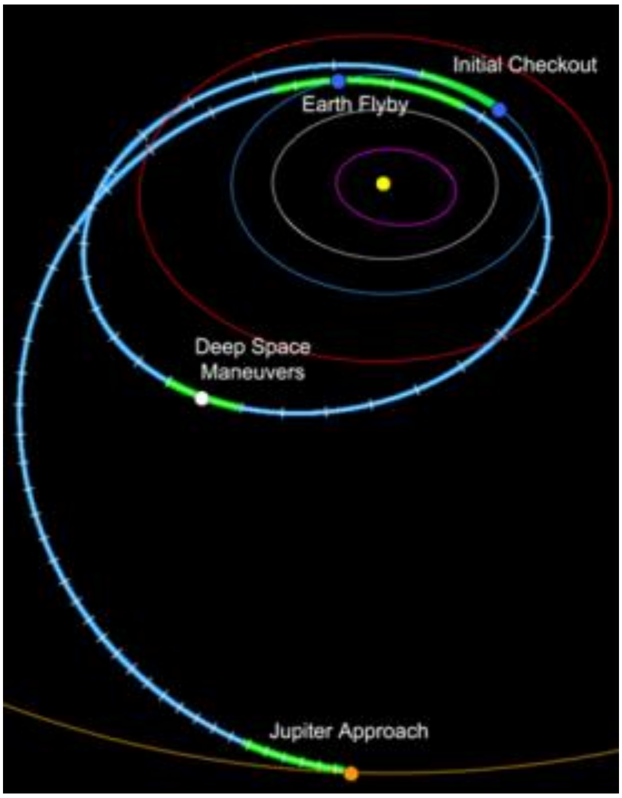
In the figure S is the sun, E is Earth, A and B are the two STEREO satellites and a plasma cloud C originating from a corona flare is approaching Earth.
According to the measurements of satellite A, angle SAC is 45°, and according to satellite B angle SBC is 50°.
At the moment of the measurement angle ASE equals 54° and angle BSE equals 48°.
Earth's orbit can be considered as a circle of radius 150 million km, the orbital radius of satellite A and satellite B is 145 million km and 156 million km, respectively.
(a) Determine distance CE and angle CSE.
(b) If the plasma cloud originating from the solar flare is moving at a speed of 2 million km/h, how long does it take for it to cover the distance SC?



3.11 The Juno space probe was launched in August 2011. Its initial elliptical orbit was designed to fly by Earth in October 2013 after a change of orbit using its rockets in August 2012. The gravitational attraction of Earth set the space probe on the elliptical transfer orbit around the Sun, where it could get close to Jupiter without requiring extra energy. (See figure.) The equation of the transfer orbit is

$$5.15x^2 + 9.61y^2 = 49.49$$

where distances are given in astronomical units.
(a) Calculate the length of the semi-major axis and the semi-minor axis of the orbit, the eccentricity of the orbit, and the distance at perihelion and aphelion. Give distances in AU.



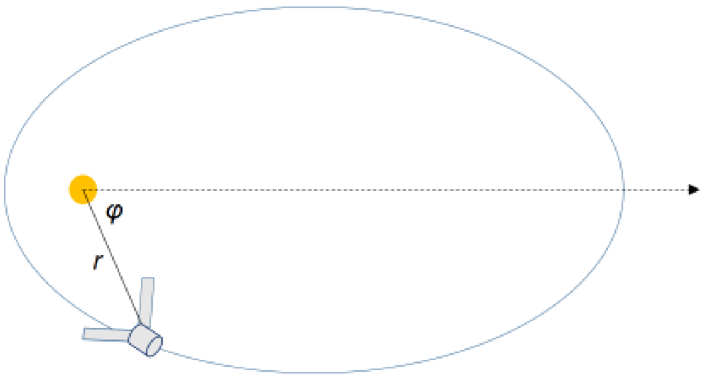
http://spacemath.gsfc.nasa.gov

(b) When did the space probe arrive at Jupiter?

3.12 As the Juno space probe travelled towards Jupiter on an elliptical orbit around the Sun, the distance from the Sun gradually increased. As Juno produces electricity with solar cells, the available power decreased with increasing distance according to an inverse square law. At the distance of Earth, the power generated by the solar panels was 12,690 W.
The distance r from the focus of an ellipse can be given as a function of the angle φ as follows:

$$r = a \cdot \frac{1-e^2}{1-e\cdot\cos\varphi}$$

The length of the semi-major axis of space probe Juno is approximately 3.0 AU, its eccentricity is approximately 2/3.



(a) Find the value of the angle φ where the generated power was one quarter of the power at a distance 1 AU.

(b) Find the generated power when the space probe arrived at Jupiter at the most distant point of its orbit.

3. Cosmic research and its tools

INCREASING THE RESOLUTION OF DETECTION

3.13 The angular resolution of an optical device is the smallest angular distance between two object points, where they can be identified as two separate points in the image formed by the device. For waves of wavelength λ entering through a circular aperture of diameter D, the angular resolution is

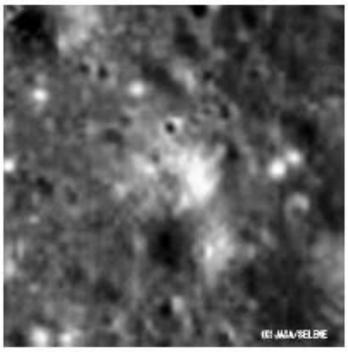
$$\alpha = 1.22 \cdot \frac{\lambda}{D} \, .$$

(a) The diameter of the mirror of the Hubble Space Telescope is 2.4 m. Find its angular resolution for the wavelength 550 nm.

(b) How far could a Sun-sized star be so that Hubble could distinguish it from a point light source (that is, detect it as an extended light source)? Can Hubble be used to see the disk of stars of the same size as the Sun?

(c) Betelgeuse (α Orionis) is 1200 times the size of the Sun and it is 430 light-years away. Can we see Betelgeuse's disk using Hubble?

3.14 The following two photos were taken of the place of Apollo-15's landing on Moon. (Around the centre of the second image the horizontal shadow of the Apollo-15 landing unit can also be seen.)
(a) Which photo was taken by the Japanese Kaguja satellite (resolution: 10 m/pixel, aperture size: about 15 cm) and which by the Lunar Reconnaissance Orbiter (LRO) satellite (resolution: 1.0 m/pixel, aperture size: 0.8 m)?



http://spacemath.gsfc.nasa.gov

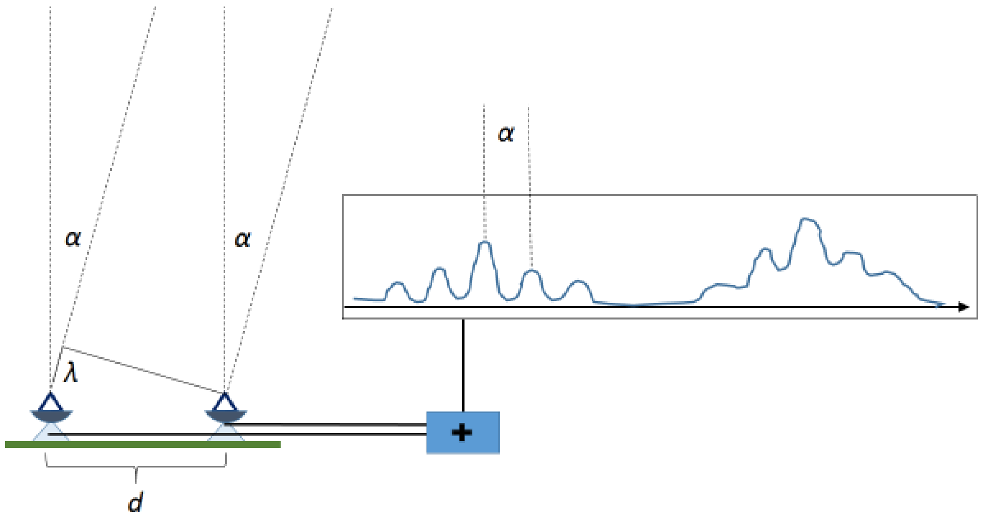
(b) The height of the orbit of the LRO satellite was 50 km. At what height above the surface of the Moon did the Kaguja satellite take the photo?

3.15 (a) Find the diameter of the antenna of a radio telescope sensitive at a wavelength of 21 cm if its angular resolution is equal to that of a mirror telescope with a diameter of 15 cm.

Remark:
A radio telescope of that size is obviously impossible to build. However, the desired resolution can be achieved using interferometry, i.e. placing two smaller radio telescopes at a distance of such magnitude.

(b) Two radio telescopes placed in the east-west direction, interconnected as an interferometer and set for receiving radio waves of wavelength 550 nm are directed at the meridian. The distance d (baseline) of the two antennas is one thousand times the wavelength.
As Earth slowly rotates, the phase difference of the signals arriving at the two antennas from a given radio source changes. As the source slowly passes through the field of vision of the telescopes, the electronic superposition of the signals received by the two antennas results in a pattern of constructive and destructive interferences.

Find the accuracy of the instrument in determining the direction of the source, i.e. find the size of angle α in the figure.



(c) How does the resolution change if the two antennas are not directed at the meridian?

(d) The figure shows the superposed signal from two different sources. In the first case, there are places of cancellation between the maxima, while in the second case the sum is non-zero at the minima. What can be the difference between the two sources?

3.16 (a) Find the maximum possible resolution of the radio telescopes operating at Onsala (Sweden) and Amherst (USA, Massachusetts State, to 2011) being 2900 km from each other on a straight line used as a very long base interferometer (VLBI) at 22 GHz frequency.
(b) Find the diameter of an optical telescope with the same resolution.

3.17 Find the maximum resolution available with a 5,000 km-baseline radio interferometer operating at 5 GHz frequency.

Solutions 3.

3.1 (a) $E = \frac{1}{2}mv^2 =$

$$= \frac{1}{2} \cdot 10000000 \cdot 18000^2 = 1.6 \cdot 10^{15}\,\text{J}$$

It is equivalent to 386 kilotons of TNT. (About 10 small nuclear bombs.)

(b) The inhabited area is only 0.8%, that is, 1/125 of the whole Earth. With two such events per year the probability is 1/62, so it happens about once in 62 years, that is, once, maybe twice in a human lifetime.

(c) A bolide arrives on an inhabited area with a probability of 1/125, which is 400 out of the 50,000 per year. In order to see it, we have to be in the appropriate 100-km^2 area. This area is 1/500,000 of the surface of the whole Earth, or 1/40,000 of the inhabited area. 400/40,000 = 100, so each year we have a 1 percent chance on average. That is, if we watch the sky in our whole life, we can see such event once every 100 years on average.

3.2 (a)

$$E = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 5 \cdot 10^9 \cdot 20000^2 = 1.0 \cdot 10^{18}\,\text{J}$$

$$\log D = \log(1.96 \cdot 10^{-5}) + 0.294 \cdot \log E$$

$$\log D = \log(1.96 \cdot 10^{-5}) + 0.294 \cdot 18 = 0.584$$

D = 3.8 km

(b) $\log D = \log(1.96 \cdot 10^{-5}) + 0.294 \cdot \log E$

$$\log 100 = \lg(1.96 \cdot 10^{-5}) + 0.294 \cdot \log E$$

$$6.71 = 0.294 \cdot \log E$$

$$\log E = 22.8$$

$$E = 6.54 \cdot 10^{22}\,\text{J}$$

It is equivalent to $1.6 \cdot 10^{10}$ kilotons of TNT.

3.3 (a) $R(m) = 0.011 \cdot 37\,000 \cdot 10^{-0.2m} =$

$$= 407 \cdot 10^{-0.2m}$$

(b) $\log R = -0.2m + \log 407 = -0.2m + 2.6$

$$m = \frac{\log R - 2.6}{-0.2} = 13 - 5\log R$$

If 200 < R < 1000, then
2.3 < log$R$ < 3
1.5 > 13 – 5log$R$ > –2
Yes, it is clearly visible.

3.4 (a) If the horizontal size of the image is 16 cm, then 1 cm is equivalent to 500 m. The largest size of the core of the comet is about 14 cm, that is, 7 km, in the perpendicular direction it is 11 cm, that is, 5.5 km. The size of the craters is 8 mm, that is, 400 m. Details with size of about 1 mm, that is, 50 m are still visible.
(b) If its path had run 2 cm to the right in the photo, then it would have missed the core. It allows for an inaccuracy of only 1 km.

Remark:
As its distance from Earth is about one hundred million km, its path had to be determined with a relative error less than one hundred millionth.

3.5 (a) $M = \rho \cdot \frac{4}{3}\pi R^3 =$

$$= 400 \cdot \frac{4}{3}\pi \cdot 3000^3 = 4.5 \cdot 10^{13}\,\text{kg}$$

(b) The change in mass is negligible compared to the mass of the core of the comet. Using the law of conservation of momentum:

$$mv = Mv'$$

$$v' = v\frac{m}{M} = 10300 \cdot \frac{362}{4.5 \cdot 10^{13}} =$$

$$= 8.2 \cdot 10^{-8}\,\text{m/s} = 2.6\ \text{m/year}$$

(c) Now

$$v' = v\frac{m}{M} = 10300 \cdot \frac{7.5 \cdot 10^8}{4.5 \cdot 10^{13}} =$$

$$v' = 0.17\,\text{m/s} = 5400\ \text{km/year}$$

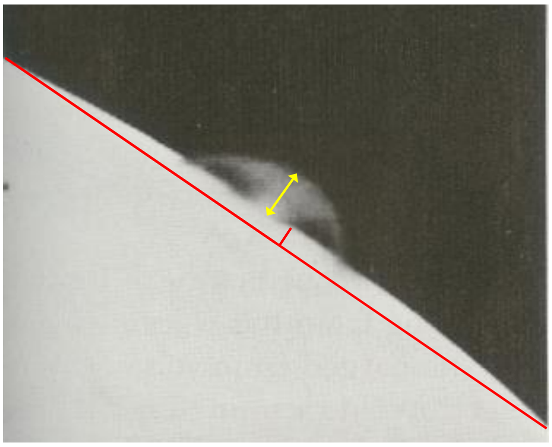
it is 108,000 km in 20 years. Earth’s diameter is only 13,000 km, so collision can be avoided.

3.6 If the width of the photos is 38 mm, then 1 mm corresponds to 0.05 light-year. In the first photo the width of the cloud is 25 mm, in the last one it is 37 mm, so the increase in the diameter is 12 mm, the increase in the radius is 6 mm. It gives an expansion of 0.3 light-years in 5 years, so the average speed is 3/50 of the speed of light, that is, about $1.8 \cdot 10^7$ m/s.

3.7 From 1977 to 2017 40 years passed.
(a) $s = vt = 40 \cdot 365 \cdot 24 \cdot 3600 \cdot 17000 =$
$= 2.1 \cdot 10^{13}$ m = 140 AU
(b) The speed of the probe is $5.67 \cdot 10^{-5} c$,
the closest star is at a distance of 4.3 light-years.

$$t = \frac{4.3}{5.67 \cdot 10^{-5}} = 76000 \text{ years}$$

(c) If the longest chord of the arc shown in the photo is 216 mm, then the height of the circular segment is 7.2 mm.



$$R^2 = 108^2 + (R - 7.2)^2$$

From this equation R = 810 mm, which is equivalent to 1815 km. The height of the cloud is 17.2 mm, that is,

$$\frac{17.2}{810} \cdot 1815 = 39 \text{ km}$$

(d) Io's mass is $M = \frac{4\pi\rho R^3}{3} =$

$$= \frac{4\pi \cdot 3.55 \cdot 10^3 \cdot (1.82 \cdot 10^6)^3}{3} = 8.9 \cdot 10^{22} \text{kg}$$

The gravitational acceleration on its surface is

$$g = \frac{GM}{R^2} = \frac{4\pi G\rho R}{3} =$$

$$= \frac{4\pi \cdot 6.67 \cdot 10^{-10} \cdot 3550 \cdot 1.82 \cdot 10^6}{3} = 1.8 \text{m/s}^2$$

$$mgh = \frac{1}{2} mv^2$$

$$v = \sqrt{2gh} = \sqrt{2 \cdot 1.8 \cdot 39000} = 370 \text{ m/s}$$

Remark:
On Earth the maximum speed is around 100 m/s even in the case of the most intense volcanic eruptions (Krakatoa, St. Helens, etc.). It is likely that the volcanic activity on Io is partly a result of phenomena fundamentally different from the phenomena on Earth.

3.8 (a) Perihelion on 4 January, aphelion on 4 July.
(b) If the greater diameter shown in the photo is 122 mm, then the smaller one is 118 mm. The average is 120 mm. The deviation from the average is 2 mm, 2/120 = 1.7%
Distance is inversely proportional to the apparent size, so the deviation is ±1.7%.
(c) The difference is

$$149\,600\,000 \cdot \frac{122 - 118}{120} \approx 5\,000\,000 \text{km}$$

3.9 (a) $r_1 = 31 \cdot 1.5 \cdot 10^{11}$ m $= 4.7 \cdot 10^{12}$ m,
$r_2 = 49 \cdot 1.5 \cdot 10^{11}$ m $= 7.3 \cdot 10^{12}$ m,
The luminosity of the Sun at distance r from it is

$$\frac{L}{4\pi \cdot r^2}$$

Due to its rotation, the incident luminosity on Pluto's surface is only one quarter of this value:

$$I = \frac{L}{16\pi \cdot r^2}$$

$I_1 = 0.35$ W/m^2, $I_2 = 0.15$ W/m^2,
Absorbed intensity = emitted intensity:

$$0.4\,I = \sigma T^4$$

$$T_1 = \sqrt[4]{\frac{0.4 \cdot 0.35}{5.67 \cdot 10^{-8}}} = 40 \text{ K},$$

$$T_2 = \sqrt[4]{\frac{0.4 \cdot 0.15}{5.67 \cdot 10^{-8}}} = 32 \text{ K}.$$

Remarks:
1. In reality temperatures are a bit higher than these, the average value is about 50 K.
2. The height of the atmosphere changes constantly, approximately proportionally with temperature. When methane gradually freezes on the surface, the height of the atmosphere decreases.

3.10 (a) In triangle ANB

ANB$\angle$ = 54° + 48° = 102°.

Using the cosine rule

$$AB^2 = 145^2 + 156^2 - 2\cdot 145\cdot 156\cdot \cos 102°$$

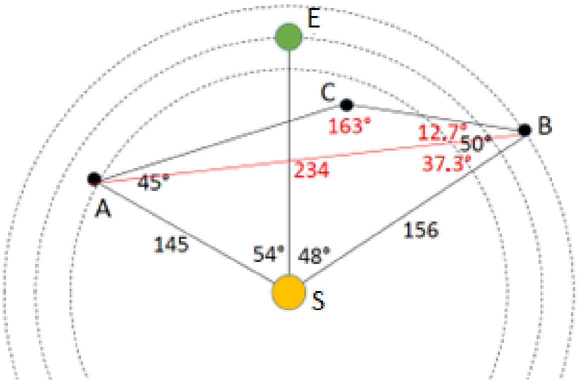
AB = 234.

Using the sine rule in triangle NBA

$$\sin \text{NBA}\angle = \sin 102°\cdot \frac{145}{234}.$$

NBA$\angle$ < 102°, so NBA = 37.3°.

ABC$\angle$ = 50° – 37.3° = 12.7°



Using the sine rule in triangle ABC

$$AC = 234\cdot \frac{\sin 12.7°}{\sin 163°} = 176.$$

Finding side CN of triangle ACN using the cosine rule:

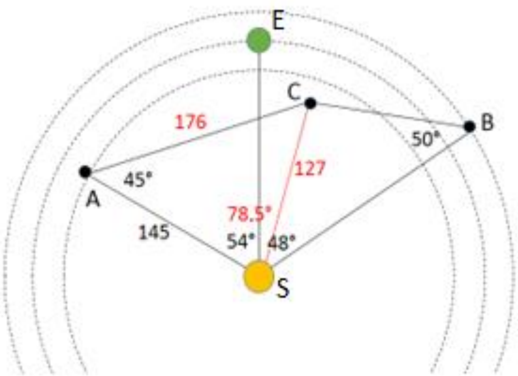
$$CN^2 = 150^2 + 176^2 - 2\cdot 150\cdot 176\cdot \cos 45°$$

CN = 127 million km.

Using the sine rule

$$\sin \text{ANC}\angle = \sin 45°\cdot \frac{176}{127}$$

ANC$\angle$ < 102°, so ANC$\angle$ = 78.5°.



So the requested angle FNC is

78.5° – 54° = 24.5°.

Finding side CF in triangle FNC using the cosine rule:

$$CF^2 = 150^2 + 127^2 - 2\cdot 150\cdot 127\cdot \cos 24.5°$$

CF = 63 million km.

(b) CN = 127 million km, the speed is 2 million km/h, so it arrives at C in about 60 hours.

3.11 (a) $\dfrac{x^2}{9.61} + \dfrac{y^2}{5.15} = 1$

$a = 3.1$ AU, $b = 2.3$ AU

$c = \sqrt{a^2 - b^2} = 2.1$, $e = c/a = 0.68$.

Perihelion: 3.1 −2.1 = 1.0 AU,
aphelion: 3.1 +2.1 = 5.2 AU.
That is, the distance of Earth and of Jupiter from the Sun.

(b) According to Kepler’s 3rd law (distances measured in astronomical units, periods in years)

$$T = a^{3/2} = 3.1^{3/2} = 5.5 \text{ years}$$

It was at the distance of Earth in October 2013.
It arrived at Jupiter half period, that is, 2 years and 9 months later, in June or July 2016.

3.12 (a) $r = a\cdot \dfrac{1-e^2}{1-e\cdot\cos\varphi}$,

Using the data of the probe

$$r = 3\cdot\frac{\frac{5}{9}}{1-\frac{2}{3}\cdot\cos\varphi} = \frac{5}{3-2\cos\varphi}$$

Power decreases to one quarter when distance is doubled. At the Earth’s distance r = 1, so

$$\frac{5}{3-2\cos\varphi} = 2$$

φ = 76°

(b) At aphelion φ = 0°, cos φ = 1.

$$r = \frac{5}{3-2\cdot 1} = 5$$

At a distance of 5 AU power decreases to only 1/25 of the original:
12690 / 25 = 508 W.

3.13 (a)

$$\alpha = 1.22 \cdot \frac{\lambda}{D} = 1.22 \cdot \frac{5.5 \cdot 10^{-7}}{2.4} = 2.8 \cdot 10^{-7}\,\text{rad}$$

(b) The angular size of the star must be greater than this value.

The diameter of the Sun is $1.4 \cdot 10^9$ m, so

$$d < \frac{1.4 \cdot 10^9}{2.8 \cdot 10^{-7}} = 5.0 \cdot 10^{15}\,\text{m} = 0.53 \text{ light-years}$$

The closest star is more than 4 light-years from us, so it cannot be resolved by Hubble.

(c) Yes, 430/0.53 =810 < 1200

Hubble's photo of Betelgeuse
(source: Hudoba Gy.'s doctoral thesis, ELTE, 2016.)

3.14 (a) The second photo, whose resolution is better, was taken by LRO.

(b) The angular resolution of LRO is in the ratio of the lens diameters, 0.8/0.15 = 5.3 times the other.

The resolution of the photo is 10 times more, so the other satellite took the photo from a height 10/5.3 = 1.9 times greater, that is, from 94 (about 100) kilometres.

3.15 (a) For a given angular resolution the required diameter is proportional to wavelength. In the case of an optical telescope, using the wavelength 550 nm

$$D_2 = D_1 \cdot \frac{\lambda_2}{\lambda_1} = 0.15 \cdot \frac{0.21}{5.5 \cdot 10^{-7}} = 57 \text{ km}$$

(b) (As it is not a diffraction through a circular hole, the 1.22 factor is not needed.)

$$\alpha \approx \sin \alpha = \frac{\lambda}{d} = 0.001 \text{ rad}\,.$$

(c) A smaller effective distance is substituted into the denominator instead of *d*, so the angle is greater, the resolution is worse.

(d) The first signal comes from a source that can be regarded as point-like, but the second source has large extension relative to the resolution of the interferometer: the signals coming from its different points do not cancel out at the same place.

3.16 (a) The wavelength of the 22-GHz radio wave is

$$\lambda = \frac{c}{f} = \frac{3.0 \cdot 10^8}{22 \cdot 10^9} = 0.014 \text{ m}$$

For the angular distance between the directions of reinforcement:

$$\sin \alpha (\approx \alpha) = \frac{\lambda}{d} =$$

$$= \frac{0.014}{2.9 \cdot 10^6} = 4.7 \cdot 10^{-9}\,\text{rad} = 1.0010''$$

(b) Using the wavelength 550 nm:

$$D = 1.22 \cdot \frac{\lambda}{\alpha} = \frac{1.22 \cdot 550 \cdot 10^{-9}}{4.7 \cdot 10^{-9}} = 140 \text{ m}\,.$$

3.17 $\alpha = \frac{\lambda}{d} = \frac{c}{fd} =$

$$= \frac{3 \cdot 10^8}{5 \cdot 10^9 \cdot 5 \cdot 10^6} = 1.2 \cdot 10^{-8}\,\text{rad} = 0.0025''$$

4. Measurements, exercises requiring independent student activity

4.1 The quote below and the explanatory figure created with the printing tools of that time are from Galileo Galilei's work *Sidereus Nuncius* (1610), in which he first gave account of the discovery of moons orbiting around Jupiter. Galileo discovered four moons, however, in this first note he mentioned only three moons.
Retrace the objects seen by Galileo that night using the freely downloadable program Stellarium. Why did he see only three moons? What could "in the first hour of night" mean?

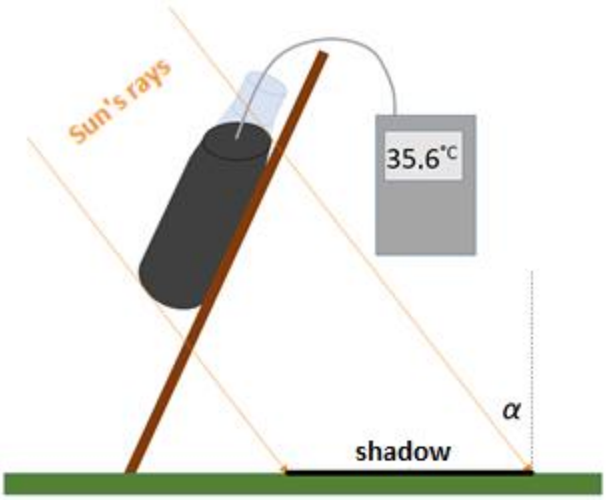
> *On the 7th day of January in the present year, 1610, in the first hour of the following night, when I was viewing the constellations of the heavens through a telescope, the planet Jupiter presented itself to my view, and as I had prepared for myself a very excellent instrument, I noticed a circumstance which I had never been able to notice before, owing to want of power in my other telescope, namely, that three little stars, small but very bright, were near the planet; and although I believed them to belong to the number of the fixed stars, yet they made me somewhat wonder, because they seemed to be arranged exactly in a straight line, parallel to the ecliptic and to be brighter than the rest of the stars equal to them in magnitude. The position of them with reference to one another and to Jupiter was as follows.*
>
> 
>
>
> *On the east side there were two stars, and a single one towards the west. The star which was furthest towards the east, and the western star, appeared rather larger than the third. I scarcely troubled at all about the distance between them and Jupiter, for, as I have already said, at first I believed them to be fixed stars.*
>
> (Translation by Edward Stafford Carlos)

4.2 Find a long street that runs in north-south direction. Step its length and determine the latitude of the two ends simultaneously using a GPS or a mobile phone application. Determine the length of your steps using a known distance. Based on the results, give an estimate of the radius of Earth.

4.3 The average intensity of solar radiation arriving on Earth (on the top of the atmosphere) is called solar constant.. The aim of this exercise is to measure the solar constant.
(a)

- Fill a small plastic bottle with thin, transparent wall with a known amount of water. (The 1-dl honey bottle known by everyone is suitable.) You do not have to fill it completely, leave space for the thermometer.
- Dye the water with black ink. (The honey bottle will be black enough from an ink cartridge for a fountain pen.)
- If the flask is fastened to a stick with a transparent adhesive tape, then it can be placed easily in the desired place and at the desired angle by sticking the stick into the ground.

- You will require a thermometer that displays tenths of degrees: a digital multimeter or a household thermometer with an outdoor sensor is suitable. (If you do not want to dip it into inky water, you can pack the sensor into household plastic foil.)
- First keep the flask with the thermometer sensor in it in the shadow until its temperature reaches the ambient air temperature.
- Then put it on the sun and read its temperature every half minute or minute. It's probably enough to continue for 15 to 20 minutes.
- Draw a line around the shadow of the dark part of the bottle on a graph paper laid on the ground. You will need the area of the shadow, so use of square paper or graph paper.
- Either measure (e.g. using the shadow of a vertical stick), or find the angle of incidence of the sun's rays at the time of the measurement.



(b) Draw a graph of temperature as a function of time. The slope of the graph is decreasing when the inky water is significantly warmer than its environment, because it gives off more heat to the environment. Therefore, use only the first, straight part of the graph. Set a straight line on it and determine the change in temperature ΔT in the corresponding Δt time interval.

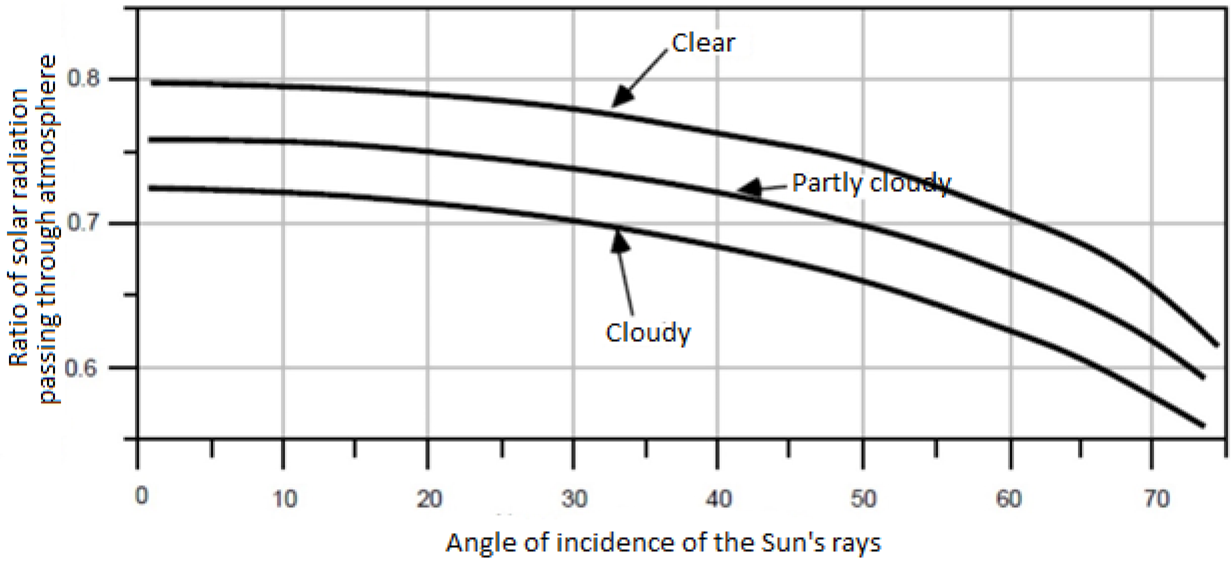
(c) Calculate the amount of heat required for the temperature change corresponding to the straight segment.

(d) Calculate the area of the shadow. Determine the cross-sectional area of the sunbeam passing through the inky water from the area of the shadow and the angle of incidence of the sun's rays.

(e) Assuming that the inky water can be considered as a black body, determine the intensity of the solar radiation on the flask from the results obtained.

(f) The received value is not the solar constant yet, as solar radiation arriving on Earth is not transmitted completely through the atmosphere. The transmitted fraction depends on how clear or cloudy, misty the sky was at the time of the measurement It also depends on the thickness of air layer the sun's rays have to pass through, that is, on the angle of incidence of the sun's rays.

Use the graph below to adjust the measurement result and determine the value of the solar constant.



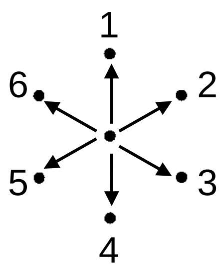
http://sbo.colorado.edu/SBO_OLD_SITE/sbo/manuals/apsmanuals/suntemp.pdf

4.4 This practice models how the radiation energy released inside a star reaches the surface of the star. The radiation cannot leave the star immediately in a straight line because the photons are re-absorbed and re-emitted.

When a photon interacts with a particle, the particle absorbs the photon, and when it re-emits the absorbed energy, a new photon starts in some direction. We can look at the phenomenon as if the same photon wormed randomly until it reaches the surface of the star.

As a simple two-dimensional model, let the star be the hexagon shown in the figures (see annex), where the points in the lattice are the particles on which the photon may be scattered. The two figures correspond to a smaller and a larger star.

Suppose the photon starts from the grid point in the middle. Use a dice (or the random number generator of a pocket calculator) to find its direction of motion.



Print the triangular grid and draw the broken line that represents the scattering according to the result of the throws. Continue to roll until the photon reaches the surface. Note the number of throws needed. Repeat it several times and/or compare the results to others'.

4.5 Helium comprises a much greater fraction of the mass of material in the Universe than the amount produced by nuclear fusion in the stars. Most of the helium was created during the so-called primary nucleosynthesis, a process that followed the Big Bang.

(a) In an early stage of the evolution of the Universe, the temperature was still high enough for the energy of thermal motion (estimated as kT, where k is Boltzmann's constant) to exceed difference $m_n c^2 - m_p c^2$ by far. Then protons and neutrons transformed into each other freely, with equal probability through reactions

$$n + e^+ \leftrightarrow p^+ + \bar{\nu},$$

$$n + \nu \leftrightarrow p^+ + e^-,$$

so approximately 50% of (ordinary) matter was proton and 50% was neutron.

The mass of a proton is $m_p = 1.67263 \cdot 10^{-27}$ kg,
the mass of a neutron is $m_n = 1{,}67493 \cdot 10^{-27}$ kg.

Give an estimate of the magnitude of temperature required for the transformation.

(b) When the age of the Universe was about 0.1 seconds, the energy of thermal motion decreased to a value around $m_n c^2 - m_p c^2$, so the reactions did not occur with the same probability in both directions, the equilibrium shifted towards protons, whose mass is smaller.

At 0.1 seconds the ratio of neutrons was only about 37%,
1 second after the Big Bang it was only about 18%.

The three figures in the first column (see annex) represent three states of the early Universe with 100 particles. Colour the circles that correspond to neutrons and leave the ones that correspond to protons blank.

(c) By the time when the Universe was 1 second old, the temperature and the particle density had decreased so much that the above reactions practically stopped. From that point on, the decay of existing neutrons played an important role. A free neutron decays to proton with a half-life of approximately 10 minutes:

$$n \rightarrow p^+ + e^- + \bar{\nu}.$$

Find the probability that a given neutron decays within a minute.

(d) The three figures in the second column illustrate this period.

Select a neutron that still exists at 1 second.
Use the random number generator of your calculator to generate a number between 0 and 1.
If the random number is less than the probability obtained in part (c), the neutron transforms into a proton (now it is white).
If it is greater, it survives the first minute, colour the same neutron in the figure of the 1-minute-old Universe. Do the same for other neutrons.

Repeat the procedure for the neutrons that survived the first minute and record the result in the figure of the 2-minute-old Universe. Similarly, create the 3-minute-old Universe.

(e) In the 3-minute-old Universe, the energy of the thermal motion decreased below the binding energy of the nuclei, so protons and neutrons formed nuclei. Because of the very high binding energy of the helium nucleus, virtually every neutron got into a helium nucleus.
In the figure of the 3-minute-old Universe, show the created helium nuclei by grouping two neutrons and two protons together. If there is a neutron left, create a deuterium nucleus with one proton.

Remark:
In the primary nucleosynthesis after the Big Bang other nuclei were also created in smaller amount, but our simple, 100-particle model is not suitable for showing these.

In your Model Universe, what percent of the mass of the matter became helium? (If you have obtained a value between 20 and 30%, the simple model is consistent with theories describing the early Universe.)

4.6 This exercise demonstrates the expansion of the Universe using a two-dimensional analogy.

(a)

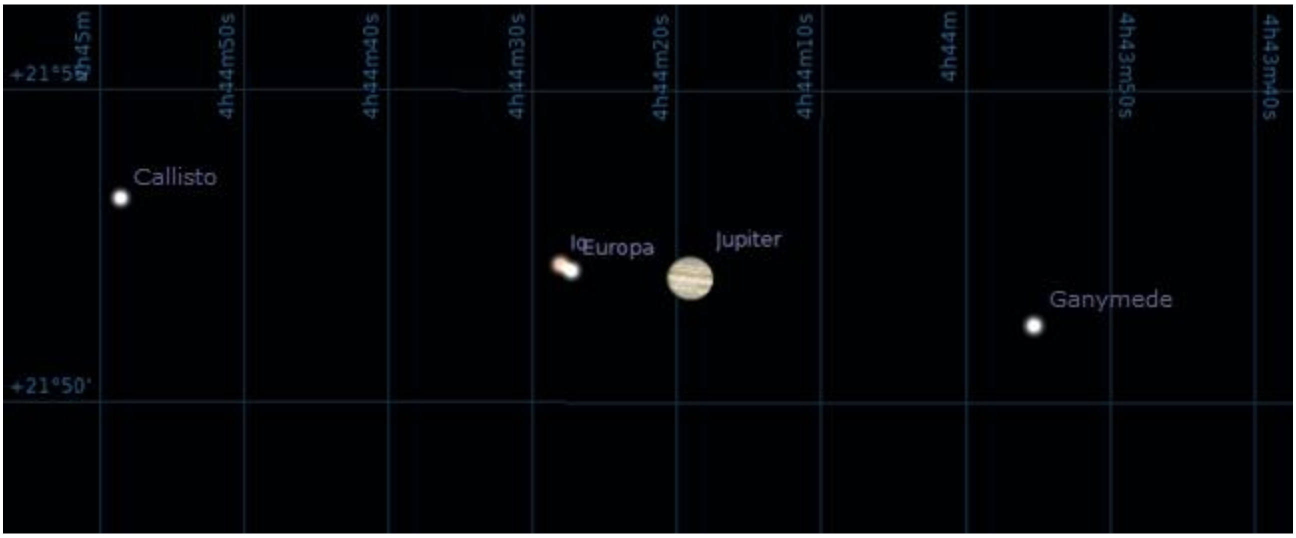
- Draw a few “galaxies” on an uninflated balloon with an alcoholic pen and number them. (No need for spiral arms, it is enough to draw dots.) Use relatively large amounts of ink, because the design fades upon inflating. Pick one of your galaxies and measure the distance of the other galaxies from this reference galaxy.
- Inflate the balloon and measure the time it takes to inflate. Close the mouth of the balloon tightly. The Balloon Universe is ready.
- Measure the distance of each galaxy from the reference galaxy again using a string (along the surface of the balloon).
- Calculate the changes in distance. Changes over time will give you the (average) speed of recession. Graph speed as a function of the (second, that is, current) distance and determine the Hubble constant for the Balloon Universe.

(b) Based on your calculations, find the estimated age of the Balloon Universe. Compare with the time of the inflation.

(c) What would be the result if you chose another galaxy as reference?

Solutions 4.

4.1 If “in the first hour of the night” refers to an observation one hour after nightfall, then Io and Europa were obscured. (It may also be that they were so close to each other that the resolution of Galileo's telescope was not high enough to differentiate them. According to Galileo's data, it is not possible to reconstruct the time of observation accurately.



4.2 In Budapest, for example, the Buda side of river Danube between Margaret Bridge and Chain Bridge runs approximately in north-south direction.
On the coastal promenade, at the foot of the Buda bridgehead of the Margaret Bridge, a latitude of 47.5145° and at the foot of the Chain Bridge a latitude of 47.4984° were measured. The difference is the central angle belonging to the arc:

$(0.0161 \pm 0.0005)^\circ = 0.0161^\circ \pm 3.1\% = 2.81\cdot 10^{-4}$ rad $\pm$ 3.1%.

The distance was found to be
2800 ± 100 steps = 2800 steps ± 3.6% long, and the length of our steps is

(62 ± 2) cm = 0.62 m ± 3.2%.

The distance, that is, the length of the arc is therefore

2800·0.63 = 1740 ± 6.8%
(= 1740 m ± 120 m).

So the radius of the circle is

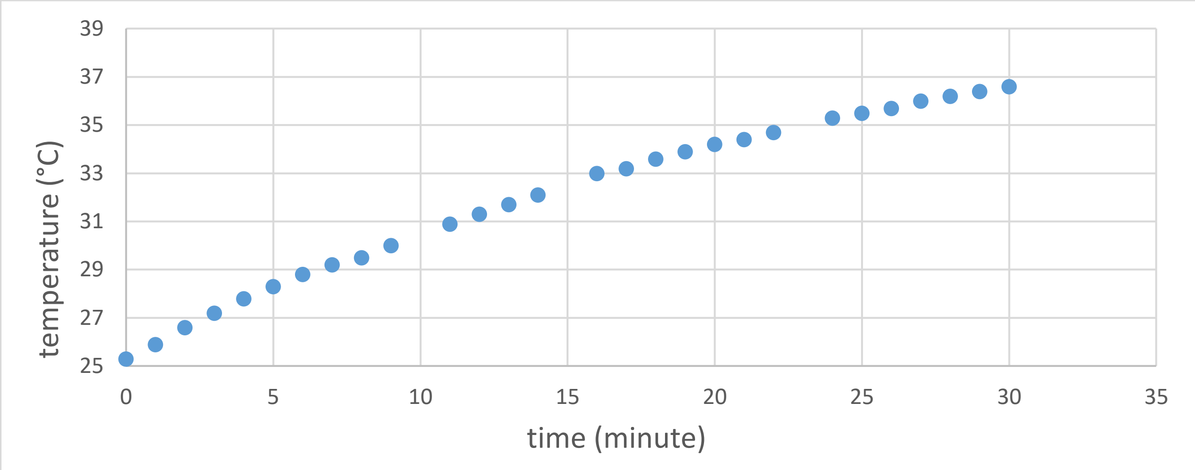
$$\frac{1740}{2.81\cdot 10^{-4}} = 6190\text{km} \pm 9.9\%$$

So our estimate for Earth’s radius is

6200 km ± 600 km.

4.3 (a) The table shows the heating of 80 g of water poured into the honey bottle on 15 August around 1 pm.

t **(minute)**	***T*** **(°C)**
0	25.3
1	25.9
2	26.6
3	27.2
4	27.8
5	28.3
6	28.8
7	29.2
8	29.5
9	30
11	30.9
12	31.3
13	31.7
14	32.1
16	33
17	33.2
18	33.6
19	33.9
20	34.2
21	34.4
22	34.7
24	35.3
25	35.5
26	35.7
27	36
28	36.2
29	36.4
30	36.6



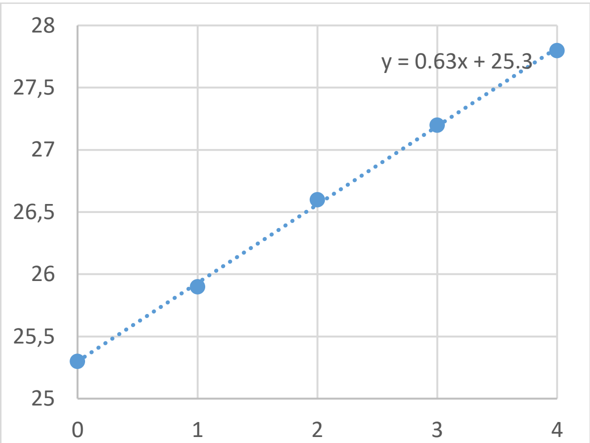
(b) The first Δt = 4-minute segment of the graph is straight enough, the change in temperature is $\Delta T \approx 2.5$°C.

(c) $Q = cm\Delta T = 4190 \cdot 0.080 \cdot 2.5 = 838\text{J}$.

(d) The area of the shadow was found to be 38.5 cm^2 by counting the squares on the square paper.
Reading from a star map, the angle enclosed by sunbeams and the horizontal is 65°, so the angle of incidence is $\alpha = 25°$.

The cross-sectional area of the beam is

$$A = 38.5 \cdot \cos 25° = 34.4 \text{ cm}^2.$$



(e) $cm\Delta T = I \cdot A \cdot \Delta t$

$$I = \frac{838}{34.4 \cdot 10^{-4} \cdot 240} = 1015 \frac{\text{W}}{\text{m}^2}.$$

(f) At the time of the measurement, the sky was quite clear, but slightly misty. Based on the given graph, the transmitted fraction can be estimated at 78%.
So the value of solar constant is

$$\frac{1015}{0.78} = (1300 \pm 100)\frac{\text{W}}{\text{m}^2}.$$

The error is primarily due to the inaccuracy of the initial slope of the graph and considering the inky water absolutely black.

4.4 The number of required throws increases fast with the number of layers.

Remark:
For example in the case of the Sun, the magnitude of time required for the energy released in the middle to reach the surface is million years. (The Sun is, of course, three-dimensional, and the process is much more complicated than this model. Among other things, because not only the size, but also the density and the temperature are important, and because the outflow of energy is not only transmitted by radiation: in the convection zone of the Sun the rise of higher-temperature gases and the sinking of lower-temperature gases is the main mechanism.)

4.5 The impact energy of particles colliding in thermal motion

$$E \approx kT = m_n c^2 - m_p c^2 =$$

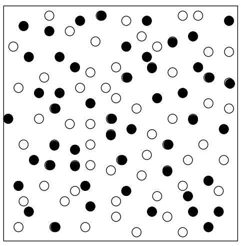
$$= (1.67493 - 1.67263) \cdot 10^{-27} \cdot (3 \cdot 10^8)^2 =$$

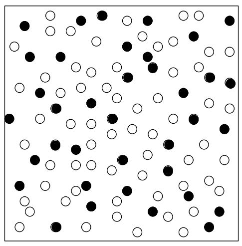
$$= 2 \cdot 10^{-13}\,\text{J}$$

$$T = \frac{2 \cdot 10^{-13}}{1.38 \cdot 10^{-23}} = 1.5 \cdot 10^{10}\,\text{K}$$

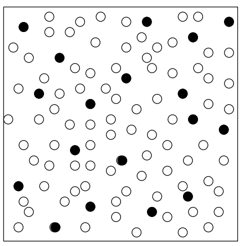
At temperature $T = 10^{11}$ K there was still equilibrium.

(b) 0.01 second: 50 neutrons, 50 protons
0.1 second: 37 neutrons, 63 protons





1 second: 18 neutrons, 82 protons

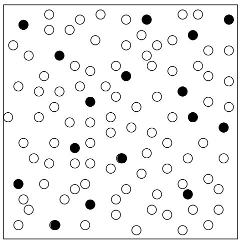


(c) After 10 minutes the ratio of remaining neutrons is 50%, so in 1 minute $0.5^{1/10} = 93\%$ remains, that is, 7% decays. This is equivalent to the statement that a given neutron decays with a probability of 0.07.

(d) For example, the 18 random numbers are

0.4538	0.8039	0.1593
0.9757	0.4912	0.0229
0.3399	0.6103	0.1113
0.2643	0.7739	0.4721
0.9454	0.2440	0.4694
0.0977	0.0008	0.5283

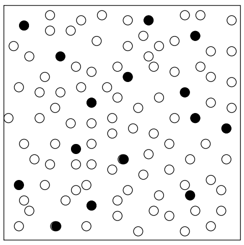
The sixth and the seventeenth values are less than 0.07, so after the first minute 16 neutrons remain.



16 new random numbers:

0.8040	0.0139	0.7914
0.1031	0.8042	0.1872
0.3738	0.1871	0.4266
0.4416	0.6985	0.6963
0.9275	0.3509	0.4525
0.5128		

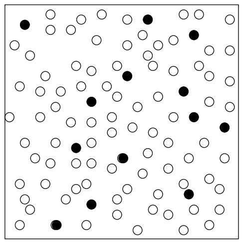
The second neutron decays, at the end of the second minute 15 neutrons remain.



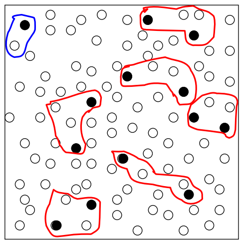
15 new random numbers:

0.4989	0.7779	0.3046
0.0424	0.7114	0.7534
0.1555	0.4160	0.0774
0.2302	0.6982	0.0630
0.5933	0.8583	0.8344

The fourth and twelfth neutrons decay, at the end of the third minute 13 neutrons remain.

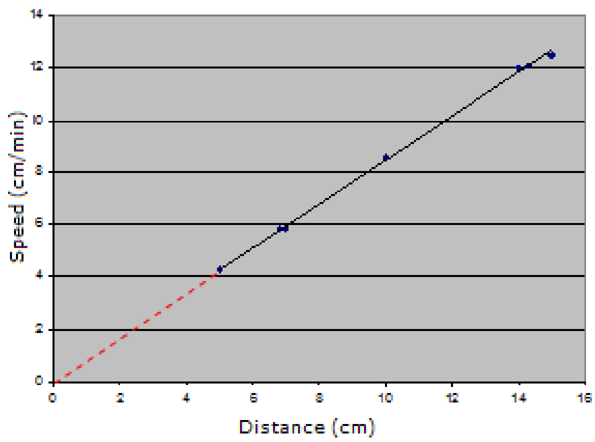


Of the one hundred particles of approximately equal mass, 6 times 4 build up helium nuclei, so the mass ratio of helium is 24%.



4.6 (a) The table contains the measurement data, the time of inflation was 1 minute.

Galaxy number	Initial distance from galaxy 0 (cm)	Final distance (cm)	Speed (cm/minute)
0	0	0	0
1	1.0	6.8	5.8
2	2.2	14.3	12.1
3	0.7	5.0	4.3
4	1.4	10	8.6
5	2.0	14	12
6	1.2	7.0	5.8
7	2.5	15	12.5



The graph is a straight line, its slope is

$$H = 0.84 \frac{\text{cm/perc}}{\text{cm}}$$

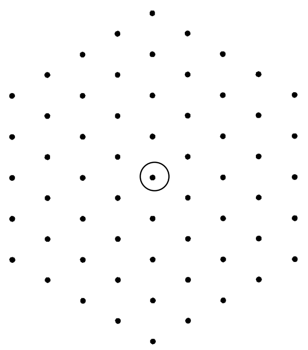
(b) The estimated age is $\frac{1}{H} = 1.2$ minutes.

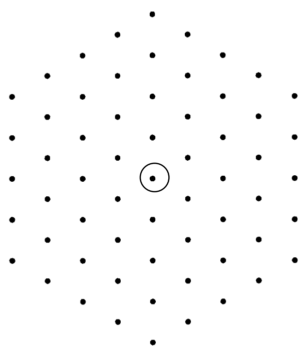
The result is different from the time of inflation because it assumes uniform expansion and in the initial state the balloon already had a finite size.

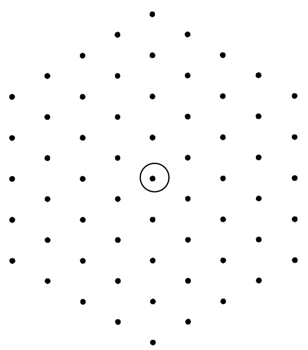
(c) We would get the same Hubble constant.

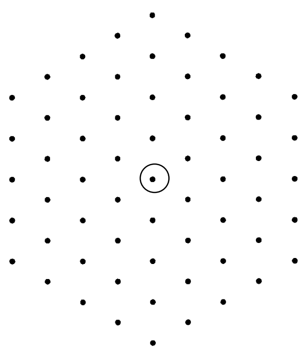
Annexes

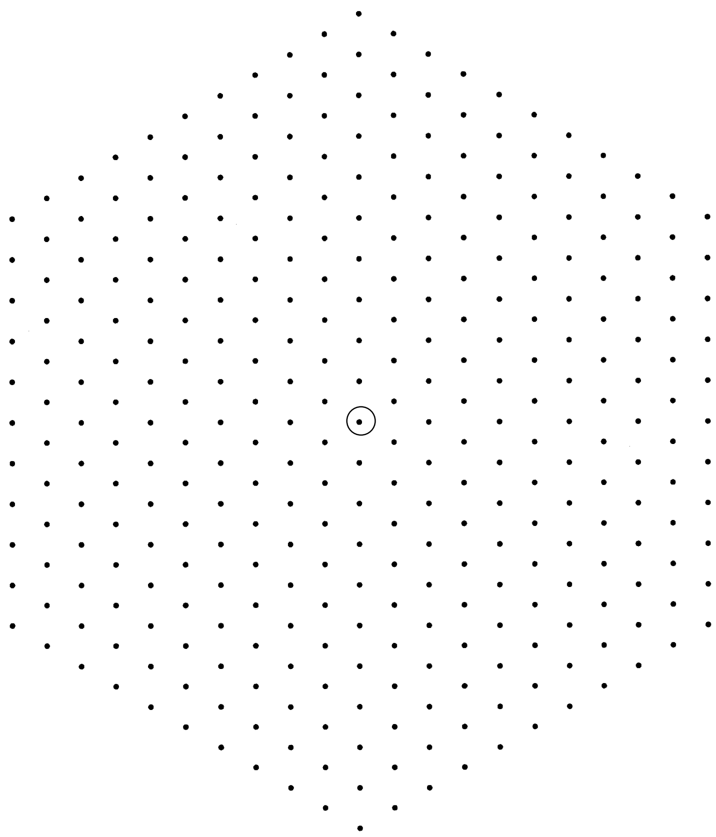
ANNEX TO EXERCISE 4.4

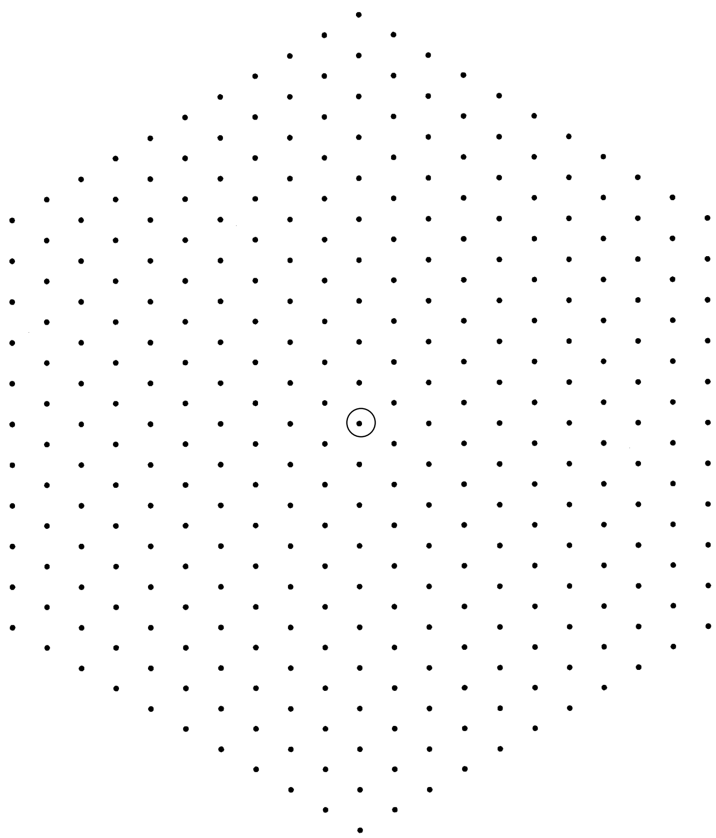






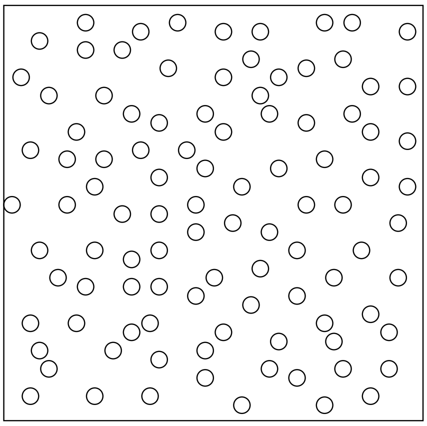




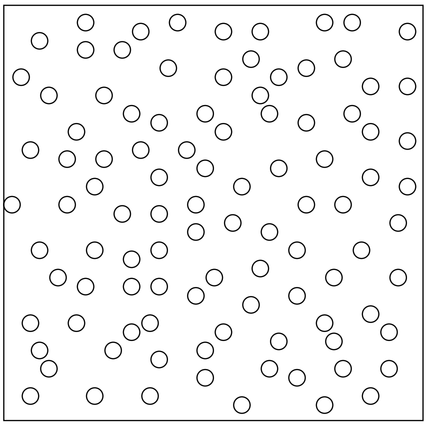


ANNEX TO EXERCISE 4.5

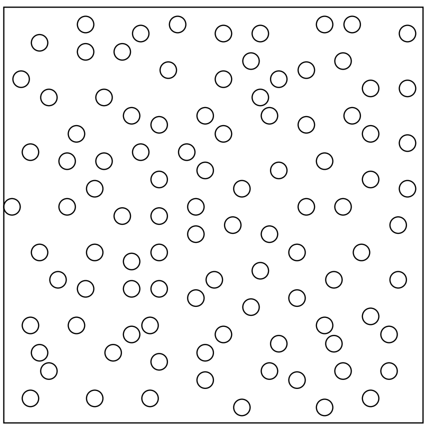
0.01 second



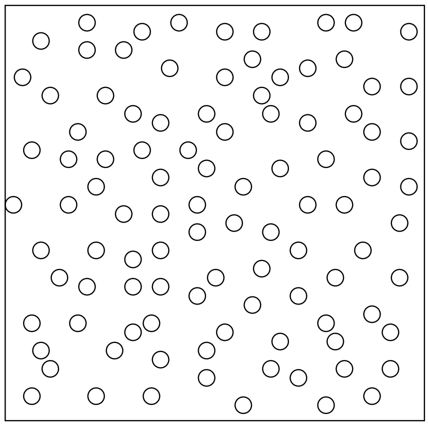
0.1 second



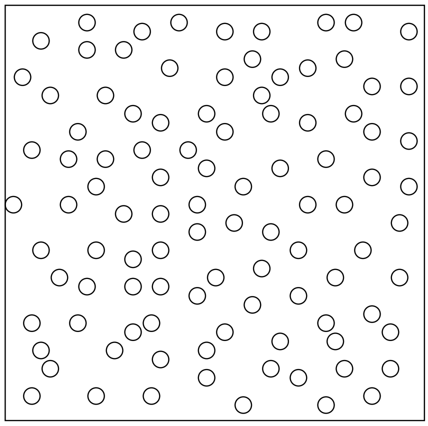
1 second



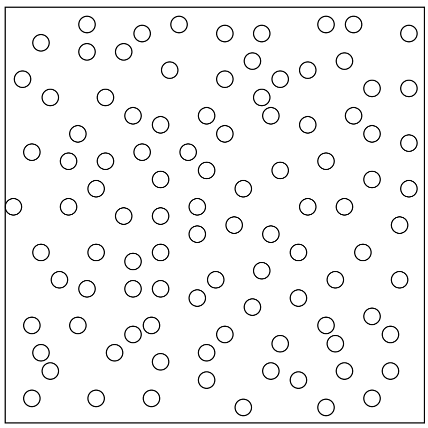
1 minute



2 minutes



3 minutes



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