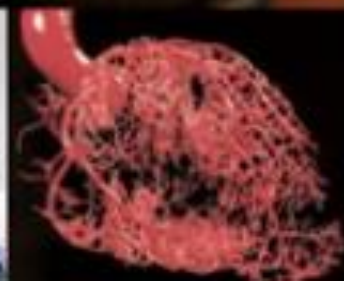




# Peter Nagy, Peter Tasnádi

## The world of fractals



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# **THE WORLD OF FRACTALS**



**ELTE DOCTORAL SCHOOL OF PHYSICS**

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## The world of fractals

The discovery of fractals shook our approach to the geometric nature of the world to the bottom: we suddenly realised that the 'customary' regular forms featured in the former image of geometry are, in fact, only a negligibly tiny segment of the living and non living material objects, while fractals grasp the genuine geometry and fundamental harmony of nature a lot deeper. Fractals are fascinating geometric structures of nature, manifesting themselves only at the very depth of our day to day experiences. Fractals provide not only the deeper understanding and simulation possibilities for the forms of nature, but revoke the filial affection in us towards the miracles and wonder of the world.

An attempt is made below to make the essence of fractals understood and to present their significance in nature and in sciences through a number of examples. In order to raise interest and to assist a better understanding, download, unpack and then view and try the computer based learning materials and simulations in the *fractalworld* folder of the ZIP file linked on [1]. This study has been made for didactic purposes and relies heavily on internet based materials, available to anybody.

### 1. What is a fractal?

As a vivid example, let's see a series of pictures taken on a seacoast with different (growing) blow-ups (Figure 1). When looking on one picture or another, you can not decide if you see the entire coastline or only a small detail of it, and hence, you do not have any reliable visual reference point to estimate the scale of the picture. The coastline is a so-called self-similar form. If you blow up a detail of a self-similar phenomenon or material object, it must be undistinguishably similar to the whole, or any other detail, therefore, if something is self-similar, its scale can not be determined.

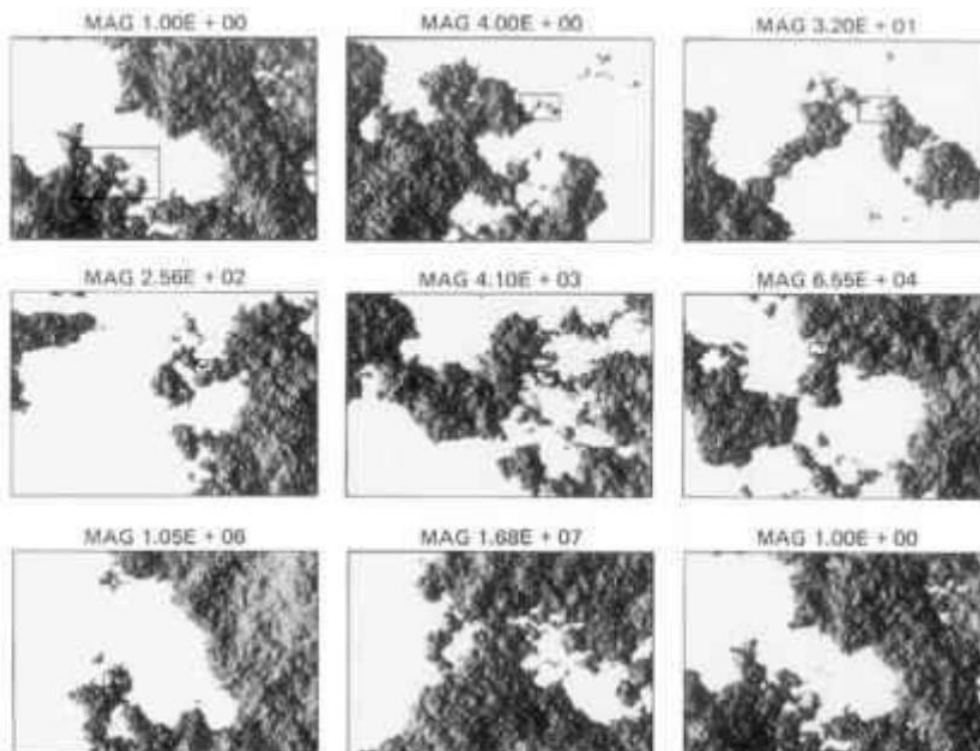


Figure 1: Self-similarity of the sea coast

(Source: [https://www.clicktocurecancer.info/natural-selection-2/images/2077\\_58\\_22.jpg](https://www.clicktocurecancer.info/natural-selection-2/images/2077_58_22.jpg))

Now put the seemingly simple question, how long is the coastline of England. Measure it with 'measuring rods' of various length (Figure 2). Four different sizes of measuring rods were used on the figure below:  $r=1$  (320 km),  $r=1/2$  (160 km),  $r=1/4$  (80 km) and  $r=1/8$  (40 km), and it was indicated, how many rods were needed to cover the coastline, obtaining values of 2880 km, 3040 km, 3840 km and 3880 km, respectively, for the length of the coastline.

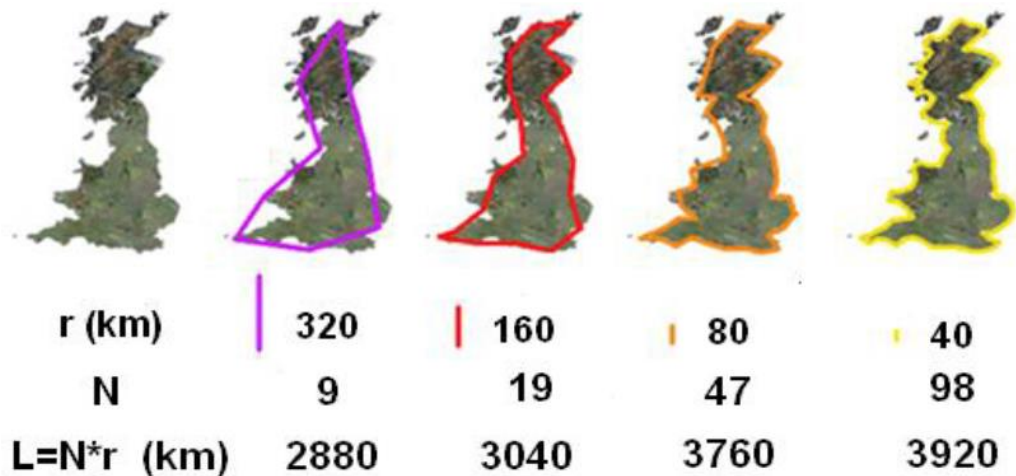


Figure 2: Measuring the length of England's coast  
 (Source: <http://fractalfoundation.org/OFC/OFC-10-4.html>)

It can be seen that the perimeter of the island of England (i.e. the length of its coastline) depends on the scale, and the length keeps on increasing as you refine the scale, what is more, it is bound to infinity! You got, after all, that England is an extremely weird geometric form: its perimeter is infinite, while its area is finite. In an analogue way, it is also possible in the case of spatial structures that a geometric object has a finite volume but an infinite surface (Figure 17). It is no mere chance that the human lungs have a fractal structure just as well, since this organ made efforts throughout evolution to have as large a surface as possible, which could be best achieved by an infinitely rugged, ramifying structure of similar forms across the scales and sizes (Figure 5.b).

Such conformations are called fractals. For the purposes of their characterisation the fractal dimension value  $D_0$  is introduced which is an extension of the customary concept of dimension. Below, fractal dimension is defined practically through measuring instructions. The fractal configuration embedded in a  $d$  dimensions geometric space is measured by covering 'cubes' with  $d$  dimensions and  $r$  size (scale) then counting the number  $N(r)$  of 'cubes' required for the cover, finally the associated  $1/r$  and  $N(r)$  values are taken in a log-log presentation with different  $r$  scales ("box-counting").

It will be experienced that the series of dots depicted this way fall onto a straight line with good approximation. The steepness of this line provides the value of the fractal dimension (Figure 3). The data of the real formations usually don't match the line exactly. In order to deem the form in question to be a fractal, it is necessary to have the picture of the  $\ln N(r) - \ln(1/r)$  function straight throughout at least two or three orders of magnitude.

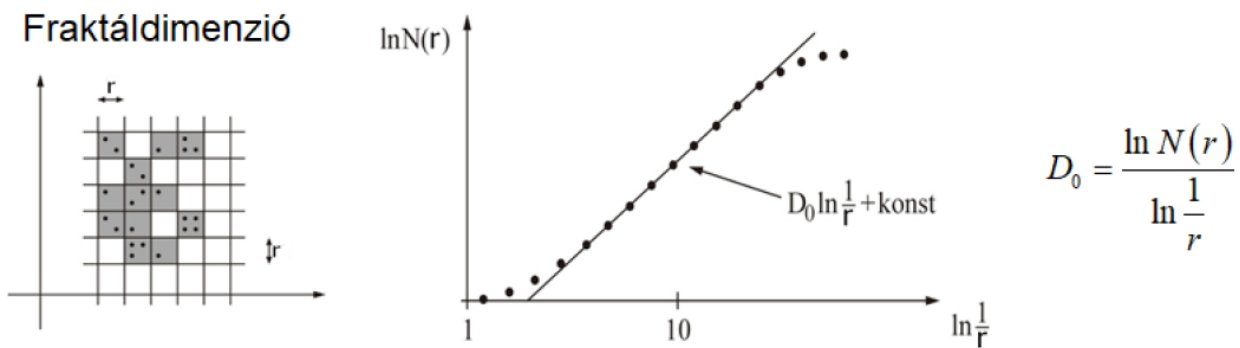


Figure 3: Measuring fractal dimensions (box-counting method)

Using the definition above the well known values (1, 2 and 3, respectively) will be obtained for the 'conventional' geometric shapes (lines, plane figures, bodies), in other words the fractal dimension for the customary forms will conform with the 'classical' dimension concept. However, when the series of measurements made on the coastline of England or Norway are depicted, fractional values 1.25 (Figure 4), and 1.52, respectively will be obtained.

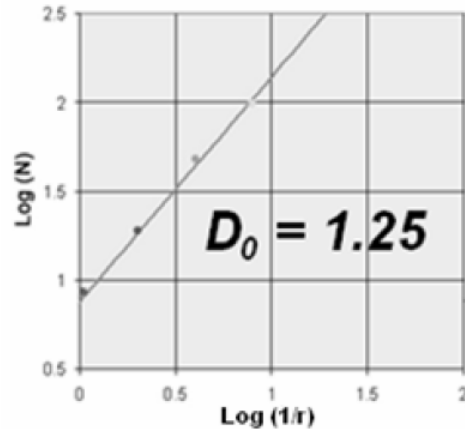


Figure 4: Fractal dimension of the coasts of England

Based on the foregoing, you can state that *fractals*:

- are infinitely complex, complicated geometric configurations,
- which are self-similar in an exact or approximate (statistical) way across a several orders of magnitude (range) scale,
- when any of their characteristics is plotted on a log-log scale (such as surface, volume or mass) as a function of their size, a line is obtained the steepness of which is the (typically fractional) fractal dimension.

Fractals surround us everywhere, especially in nature. Among other, the mountains, the trees, the lightning bolts, leaves, fjords, snow flakes and clouds are all fractal forms. These naturally occurring fractal shapes please the eye and are found to be more exciting and beautiful than the simple and rigid geometric figures. View the [1] downloadable material, the video entitled [fractals in nature.mp4](#).



Figure 5a: Fractals in nature (mountains, trees, leaves, lightning, rivers, snow flakes, clouds)





Figure 5b: Fractals in nature (plants, snails, shells, crystals, blood vessels in the lung and in the heart)

The surface of the Moon is the best known fractal (across a 12 scale of different orders of magnitude), with a fractal dimension of  $D_0 = 2.2$  (see the downloadable material [1] for the video [moon.mp4](#)).

Since fractals are fundamental structures in the world, they can be created simply, in a straightforward way.

### 1.1. Let's make fractals

Below, fractal pellets are made of paper in a simple way and the measuring method of the fractal dimension by box-counting is also illustrated. Let's take two simple A4 size papers and make it lesser and lesser in size by halving all the time, as seen on Figure 6! Indicate the smallest of the series with 1, and the largest one (Size A4) with 7. Thus, the members of the series have a mass of 1, 2, 4, 8, 32 and 64, respectively, when the mass of the smallest piece of paper is considered to be the unit (in the case of paper sheets, the mass is proportional with the surface, i.e. the square of the linear size). Crumple each sheet into the shape of a ball by pushing, rolling, kneading them strongly, but not too extensively, then measure their diameter by taking the average of several measurements for each ball!

Arrange the data into a table as follows. Indicate the mass of the pellets (paper sheets) in Column 1 of the table by taking the mass of the smallest (No. 1) paper sheet as one unit, and put the logarithm of this value into Column 2. Introduce the measured diameter of these paper pellets into Column 3, in Column 4 the diameters relative to the diameter of the smallest ball, and finally the logarithm thereof in Column 5 (for instance, the data set obtained this way can be found among the downloadable materials [1] as the [fractdim table.xlsx](#) Excel spreadsheet). In order to determine the fractal dimension the linear size and mass need to be plotted on a log-log scale, that is Column 2 and 5 are needed from the table. A line can be fit onto the data set of the obtained value pairs using for instance the *principle of least squares* (this can be easily achieved using the [lsm.xlsx](#) Excel spreadsheet in the [1] downloadable materials). The steepness of the example on the picture (i.e. coefficient of  $x$ ), in other words the fractal dimension is approximately  $D_0 = 2.53$ , which is a good match with the value of 2.5, obtained from the so-called *Flory-theory*. This theory describes the so-called *self-avoiding random walk* configurations mathematically. This term speaks for itself, and fits the paper pellets and thread balls perfectly.

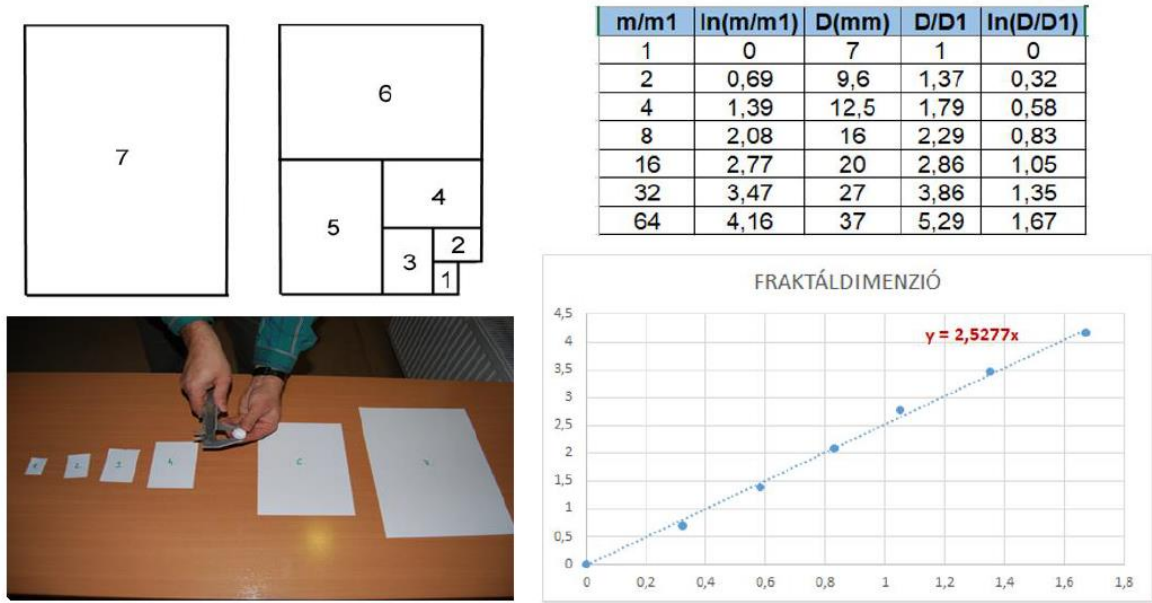


Figure 6: Making paper pellets and measuring fractal dimensions

Fractals can be created even in a simpler way. Housewives keep on folding and stretching dough when kneading it, since the best mixing procedure is achieved by the recursive algorithm of stretching and folding, resulting in a typical fractal nature. Various colour plasticine rods can also be folded and stretched in a similar way to obtain spectacular patterns (Figure 7).



Figure 7: Fractal generation by stretching-bending algorithm (dough kneading and plasticine)

People are pleased when shaping wonderful fractal structures by their own hands, stirring paint pots or using the so called marbling technique (see downloadable materials [1] for the [marble.mp4](#) video) (Figure 8)



Figure 8: Making fractals by stirring paint pots and the marbling technique

In spite of their irregularities, fractals are miraculous and reflect the deep harmony of the material world in our minds, no wonder that they have been incorporated into arts quickly, giving birth to a branch of art working with novel means.



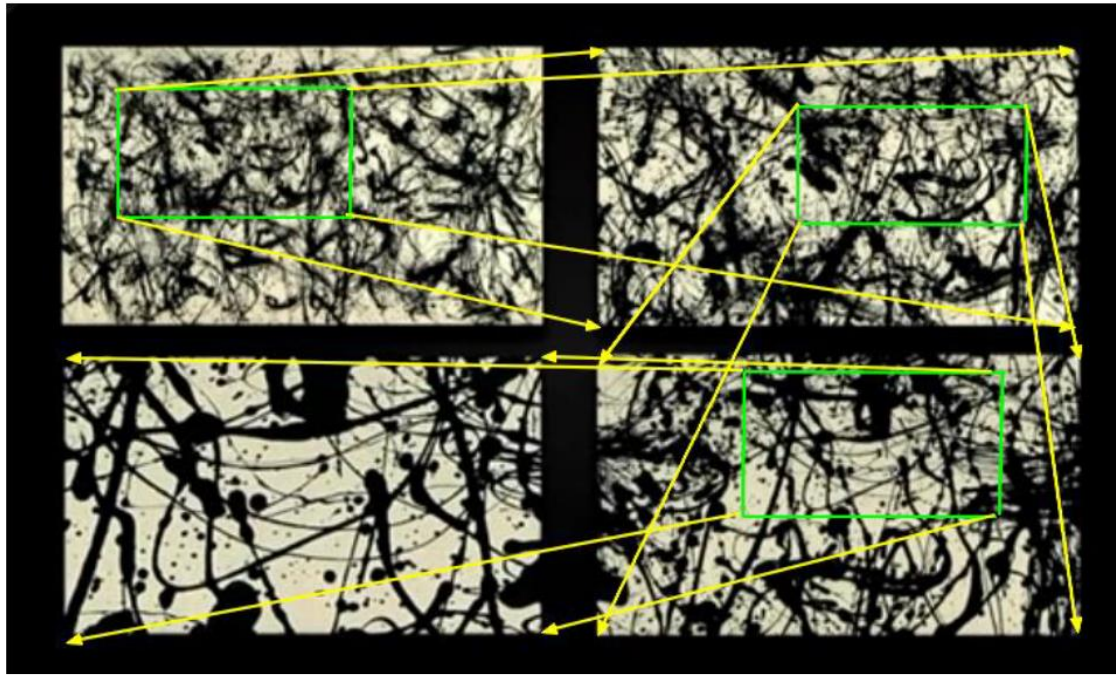


Figure 9: Self-similar fractal nature of Jackson Pollock's paintings with a series of blow-ups

The self-similarity fractal properties of the paintings by Jackson Pollock (maybe the greatest American painter of the 20<sup>th</sup> century) made using a specific technique are illustrated on Figure 9 above (we will get back to here for a couple of thoughts in Chapter 4).



Figure 10: Fractal mandala and surface burning technique

Digital art takes us to the miraculously exciting world of the fractals with the help of computer programmes (see for instance a fractal mandala by Alexa Szlávic on the left hand panel of Figure 10). Numerous programmes exist today with which anybody is able to create wonderful unique fractals on a computer with more or less effort and time. As an example, see the free *Apophysis* programme illustrated on the video entitled [Apophysis 3d fractals.mp4](#) found among the downloadable materials [1].

Beside the trickling technique applied by Pollock, high voltage surface burning techniques represent another exciting manual opportunity to create fractals (right panel on Figure 10). View the videos [fractal digital art.mp4](#), and [fractal burning art.mp4](#) among the downloadable materials [1]. A real measure for popular attention is the fact that fractals gained ground in the fashion industry [2].

## 2. Deterministic fractals

It follows from the self-similarity property that the simplest way of creating fractals is the application of *recursive algorithms*, since in such cases generation takes place according to clear rules and the configurations obtained this way are called therefore *deterministic fractals*.

First have a look at the recursive algorithm of the so-called *Koch-curve* fractal (Figure 11). Take a regular (equilateral) triangle, reduce each side by three, and draw further equilateral triangles

on each middle third section. Apply further 'outcrops' on the triangle sides thus obtained, continuing the operation *ad infinitum*. Albeit consisting of straight lines only, the curve will start to be more and more similar to a snow flake (therefore it is also called the snow-flake curve.)

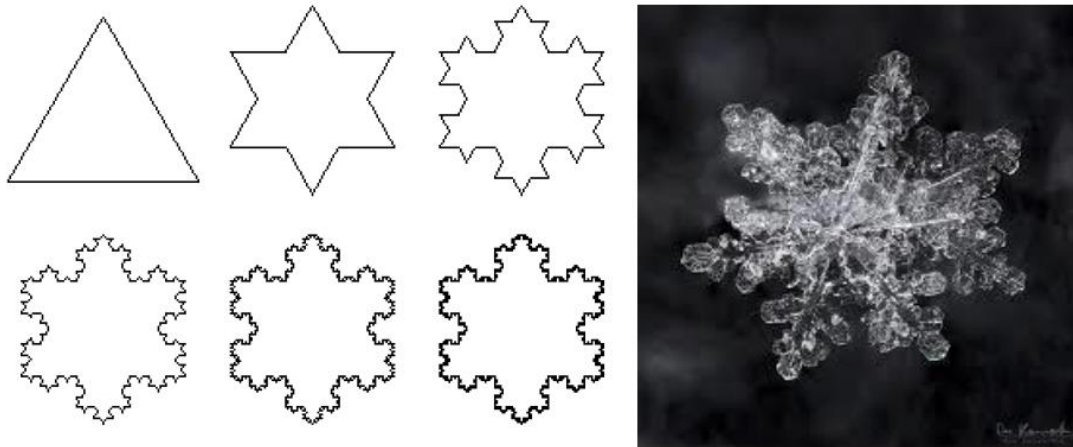


Figure 11: Generation steps of a Koch-curve (snow flake) and a real snow flake

It can be seen that in all steps of the generation process of the Koch-curve the perimeter will be incremented 4/3 fold, thus tending towards infinity, while the area remains finite, that is a fractal is received. Use the box-counting method to determine the fractal dimension. The scale of  $r$  (element size) is reduced by three in each step, that is  $r = (1/3)^n$  after steps  $n$ , while the  $N$  number of elements needed for coverage is increased four folds in each step, that is  $N(r) = 4^n$  after steps  $n$ . Hence, the fractal dimension will be as follows:

$$D_0 = \frac{\ln N(r)}{\ln \frac{1}{r}} = \frac{\ln 4^n}{\ln 3^n} = \frac{n \cdot \ln 4}{n \cdot \ln 3} = 1,262$$

As a second notable example, Figure 12 shows the generation steps of the *Sierpinski-triangle* (as well as the picture of a shell with similar patterns).

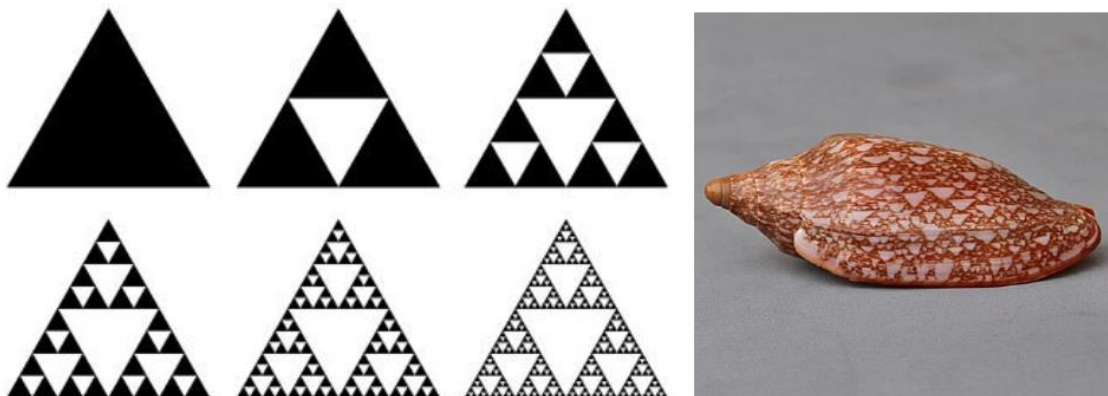


Figure 12: Generation of the Sierpinski-triangle and a shell with similar patterns

The so-called *Cantor-set* fractal is generated by the following recursive algorithm: leave the two extreme piece of  $r$ -proportion ( $r < 1/2$ ) of the sections at the beginning of any of the steps, and (the proportions may differ on the left and right) and remove the remaining piece in between (Figure 13).



Figure 13: The first four steps of Cantor-set generation at  $r = 2/5$

Based on the definition above it can be easily realised that the fractal dimension of the Cantor-set is:

$$D_0 = \frac{\ln 2}{\ln \left( \frac{1}{r} \right)}$$

that is, for instance in the case seen on Figure 13  $D_0 = 0.756$ .

An important method of fractal generation is the 'projection' of fractals, that is their *direct (Cartesian) product*. Do not get frightened of this awkward looking mathematical expression, it means a construction easy to illustrate. See for instance Figure 14 showing the first four steps of creating a *Cantor-thread* fractal. In horizontal direction, the Cantor-set recursion described earlier on takes place, while perpendicular to it, in vertical direction the filling is continuous and therefore a linear (thread-type) structure is created.

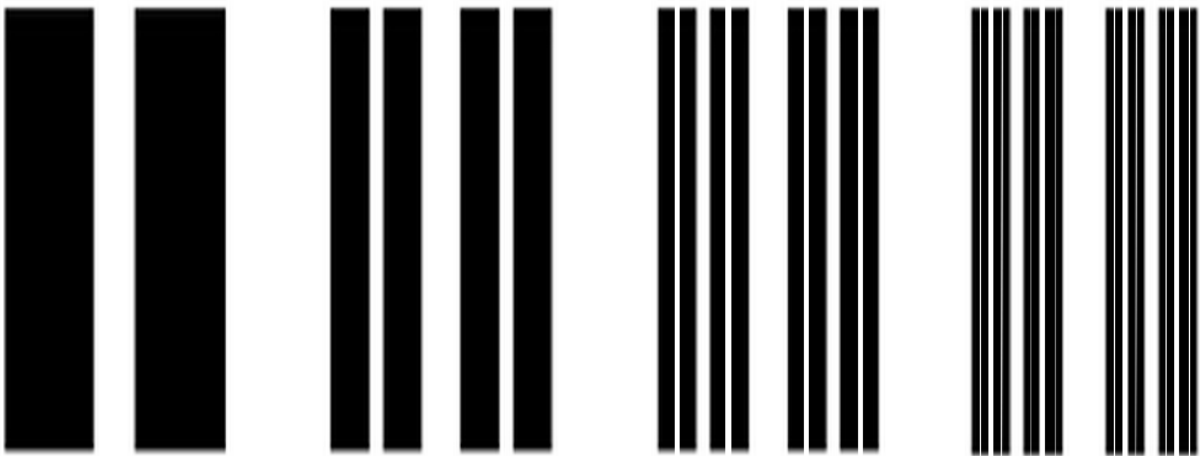


Figure 14: Cantor-thread fractal generation using direct product construction

The dimension of the conformation  $D_0$  obtained with direct product will be derived from the sum of the dimension of the two initial forms:  $D_0^{(1)}$  and  $D_0^{(2)}$

$$D_0 = D_0^{(1)} + D_0^{(2)}$$

In the case of the structure seen on Figure 14, if the Cantor-set algorithm with the parameter value  $r = 2/5$  is applied horizontally, the dimension of which, seen before, is 0.756, while lines are remain continuous vertically, with their obviously being 1, the fractal dimension of the Cantor-thread form obtained will be  $D_0 = 1.756$ .

The geometric form constructed by the direct product of the two Cantor-sets is called the *Cantor-cloud* (Figure 15).

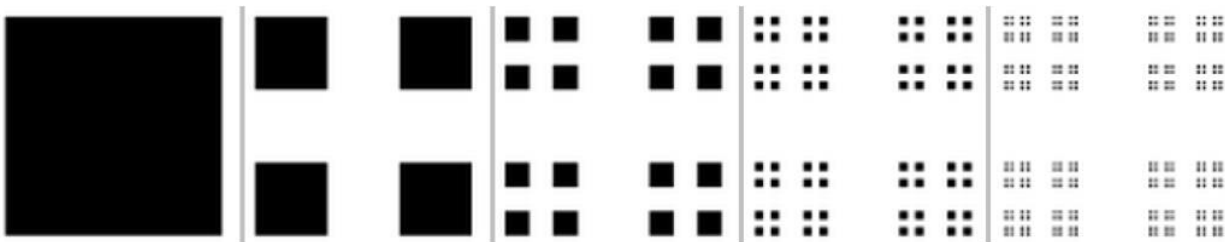


Figure 15: Cantor-cloud constructed by the 'collation' (direct (combinatorial) product) of the Cantor-set



If a Cantor-thread algorithm is used on Figure 15 which has a parameter value of  $r = 1/3$  both vertically and horizontally, how much will be the fractal dimension of the Cantor-cloud obtained?

As shown, fractals embedded in  $d$  dimensions space are measured by covering them with  $d$  dimensions  $r$  sized 'cubes', thus the volume of the fractal will be:

$$V(r) = N(r) \cdot r^d r^{-D_0} \cdot r^d = -D_0.$$

For the 'common' or 'thin' fractals, discussed so far  $D_0 < d$ , in other words their volume will disappear in the boundary case  $r \rightarrow 0$  (non space filling forms, null degree sets). There are forms, however, for which  $D_0 = d$ , therefore their volume is  $V$ , yet are strongly articulated and self-similar, i.e. are of fractal nature: *fat fractals*. For instance, you get a *fat Cantor-set* when taking out the middle  $r_n$  size piece from a section in step  $n$  (Figure 16).



Figure 16: The fat Cantor-set

As a last example, the generation of the so-called *Menger-sponge* is discussed below.

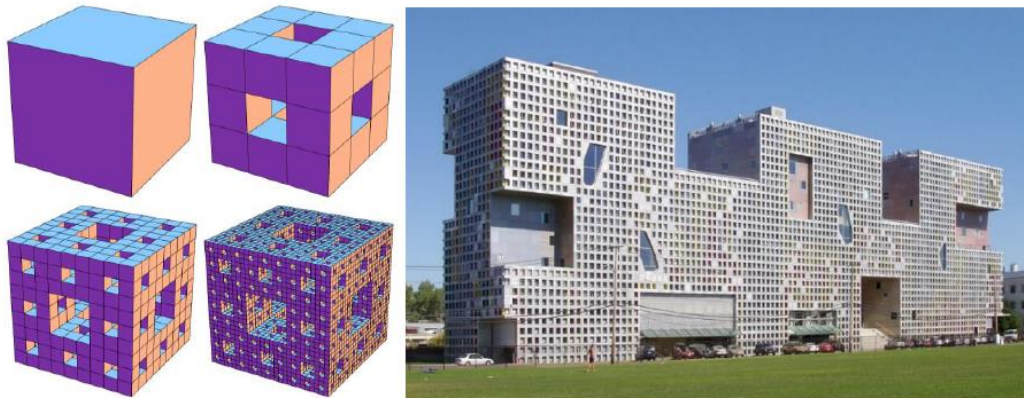


Figure 17: Recursive reconstruction of the Menger-sponge and the Simmons Hall in MIT Cambridge

The sponge is generated according to the following recursive algorithm (Figure 17 left panel): you start with a cube, which is divided into  $3 \times 3 \times 3$  parts and the cross-like form consisting of seven (!) little cube in the middle, then the remaining cubes are divided up in all subsequent steps and take the middle out. When continued up to infinity, it can be concluded, that the volume of the cubes tends to zero, since  $7/27$  part, i.e. about 26 per sent of the cubes is removed in each step. After 10 steps, only  $(20/27)^{10}$  part of the original cube will be left over, accounting roughly for 5 per sent. On the other hand, the surface of the form grows by each step, heading to infinity. Buildings with small mass and large surface can be constructed this way.

A practical and widely used way to specify and generate regular fractals with computers and recursive algorithms is the so-called *L-system declarative language* method. The method developed by A. Lindenmayer can be used to describe the steps of a *turtle graphics* easily. The symbols carry the direction of movement and length of steps of the pen. The [l-system.swf](#) Flash program (you need an Adobe Flash Player to run it) found in the downloadable materials [1] provides an excellent opportunity to create a number of well known and spectacular fractal shapes (Figure 18). You will better understand the L-system language by reading the L-system Method item in the program.

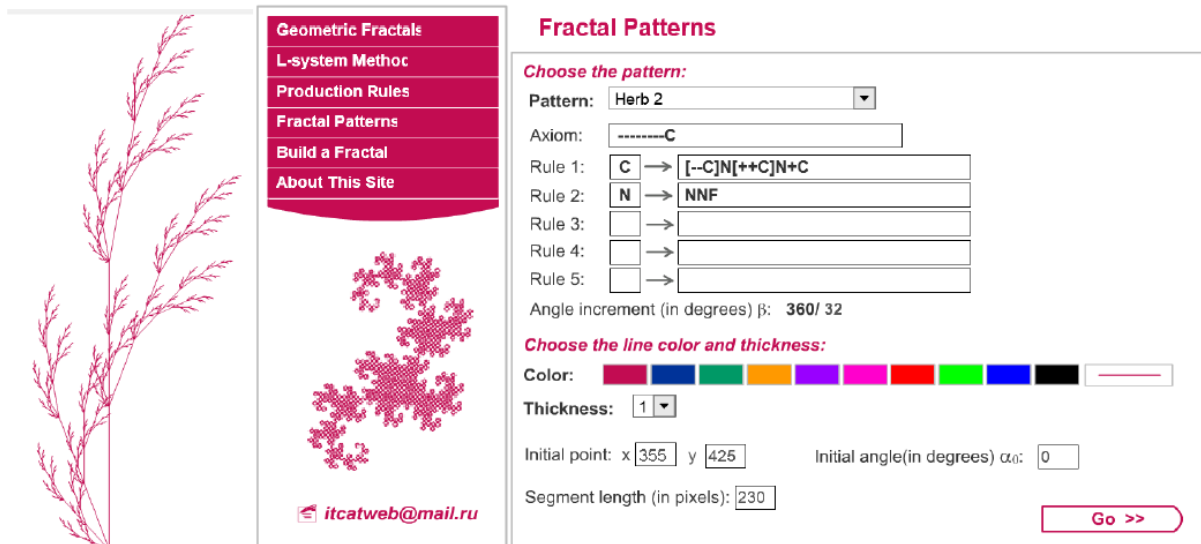


Figure 18: Screenshot of a fractal plant and L-system generator

### 3. Stochastic fractals

Another important group of fractal generating procedures consists of *random iteration algorithms*, which result in the so-called *stochastic fractals*. Somehow random moments are inserted into these algorithms: either the initial condition is generated randomly, or the value of the given parameter(s) is selected (changed) arbitrarily, or random bifurcations are inserted into certain steps. Examples for all the three possibilities are given below.

In the so-called *percolation models* each cell element on a one (two or three) dimensions grid has two states, and besides, each element also hold an inherent property. For instance, in the water permeability percolation models of soils the cell elements are soil segments of a given size with “wet” and “dry” as the two possible states. The inherent property is, whether the given segment is “porous” (“permeable”), or “non-porous” (“impermeable”). The inherent property of the cell elements is generated randomly: each element is given one of the values with a probability factor of  $p$ , and the other with a probability  $(1-p)$ . Next, an arbitrary initial state is given: for instance, all elements on the uppermost row of the soil segments is set to “wet”, all the others to “dry”. The following will takes place with each time step of the algorithm: if the dry and porous cell has at least one wet neighbour, it will become wet, non-porous cells remain dry. The first three time steps are seen from a run on Figure 19 (10\*10 field,  $p = 0.7$ ). Light blue boxes are porous, whites non porous, and black cells are seen as wet (anything which is not dark blue, is dry).



Figure 19: The first three time steps of generating a percolation fractal

The function principle of coffee machines, for instance, can be understood on the bases of the percolation models (in such a case, hot water vapour ‘percolates’ through the ground coffee beans from the below to the top, dissolving flavour substances from the coffee and distilling in the upper tank having passed the filter).

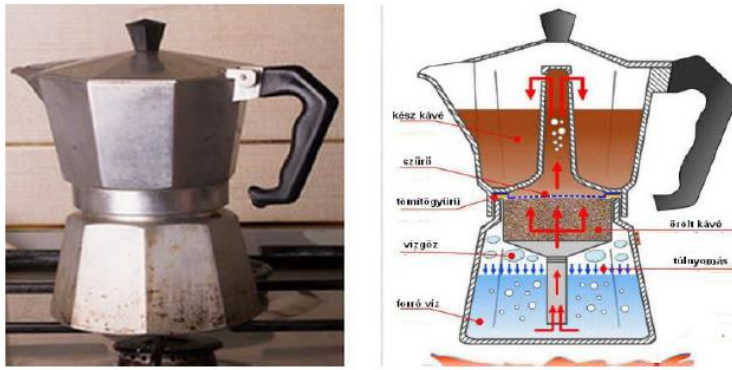


Figure 20: Operation of a coffee machine based on the percolation model

As seen above, Cantor-clouds can be created by projecting two Cantor-sets through their direct product. The class of the fractal forms thus obtained can be widened by selecting the  $r$  ratio of the right and left sections to be retained on both sides with different values and randomly from a given interval in each of the steps.

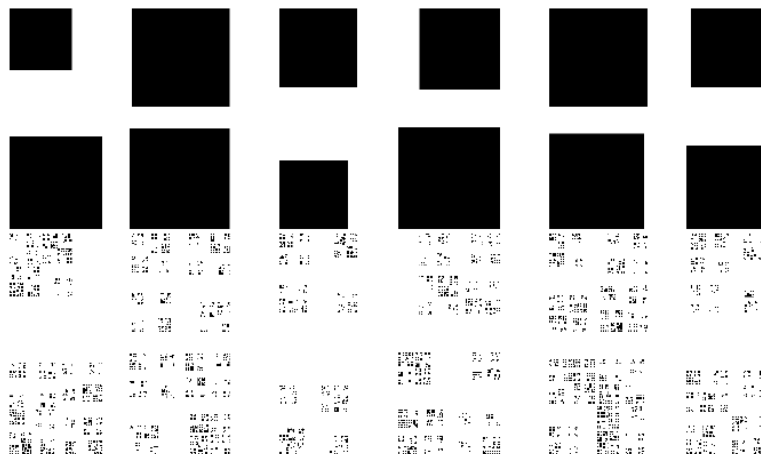


Figure 21: Generating stochastic Cantor-cloud fractals

Figure 21 shows the forms obtained after steps 1 and 10, respectively, of three different runs. Such fractals are used for encrypting images for instance.

The turtle graphics mentioned at the end of Chapter 2 can easily be rendered stochastic: the movement of the pen is directed by certain pre-determined rules and one of them is selected randomly each time (all rules are associated with a given selecting probability), and the selected rule will direct the pen in the step concerned.

### 3.1. Chaos games

Take a so-called *chaos game* played using the random recursive algorithm below:

- Select randomly a point in the inside of the (regular) **ABC** triangle.
- (\*) Roll the dice.
- Connect the selected point with
  - corner **A**, when the 1 or 2 comes out on the dice,
  - corner **B** when the 3 or 4 comes out on the dice,
  - corner **C**, when the 5 or 6 comes out on the dice,
- the new current point will be the bisecting point of the connecting section,
- jump to (\*).

(Note: it is not necessary to have a regular triangle.)

The algorithm (Java script) is found in the *chaos\_game* folder of the downloadable materials [1], it is absolutely recommended to try it (can be started using the [index.html](#) file).



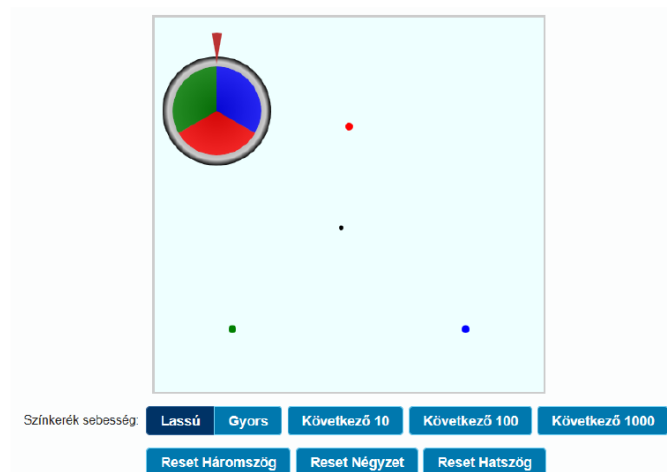


Figure 22: Screenshot of the computer aided implementation of the chaos game

In the initial state of the playing field red, blue and green coloured control points placed on the tips of the regular triangle, a black coloured current starting point in the middle of the triangle (it could be anywhere else in the inner side of the triangle), and a coloured rolling wheel with the pointing marker are seen (Figure 22). The wheel can be turned by clicking on it and the colour selected by the pointer randomly will be the control colour, while the new current point will be the bisecting point of the section between the previous (initially black) current point and the control point in the selected colour. It will appear in the selected colour and the operation is repeated over and over during the game.

Buttons under the playing field can be used to select the speed of the draw operation (slow or quick), the number of operations (and points displayed) in any one step (10, 100, or 1000), as well as the geometric figure (triangle, square, or hexagon).

In the course of the game the remarkable and interesting development can be observed (Figure 23), that the set of points will be quite similar all the times to the well known Sierpinski fractal (Figure 12), which is generated by deterministic algorithm, while this game is based on a random (stochastic) algorithm. To put it more formally, the Sierpinski fractal would contain the series of points obtained over any runs in the game with a probability of one, or, in the terms of chaos theory, the trajectory (path) attractor describing the movements and colour of the current point will be the Sierpinski-fractal itself.

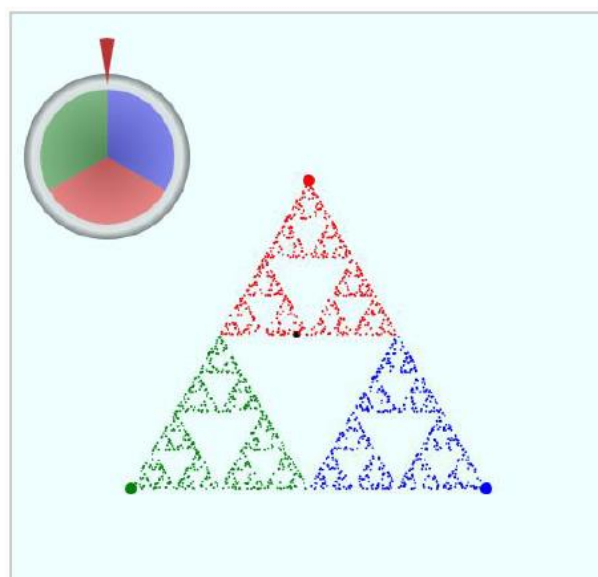


Figure 23: The dot set form developed in the course of the chaos game

How can it be explained why the set of points over the game appear always in the Sierpinski fractal shape?

Even more exciting discoveries are made by little variations of the game! The simulation entitled [chaos\\_game.ggb](#) in the downloadable materials [1] is recommended for this purpose. It implements a generalised version of the chaos game in the GeoGebra programming environment. Therefore, the freely downloadable GEOGEBRA programme must be installed for running it (for instance, GeoGebra Classic 5, or 6 on the site [3]). This implementation provides two kinds of variabilities. On one hand, the basic structure can be changed: using the upper slide the number of the control tips can be set to between 3 and 6, or even drag these corners to any place using the mouse cursor, in other words the form of the figure can be custom made. On the other hand, the lower slide can set which point of the section between the current point and the corner point selected in the given step should be the new current point (i.e. the ratio from the corner point). These two variable options provide the opportunity (and pleasure) of computerised experimentation. Detailed discussion of such opportunities are beyond the scope of this study, but they offer really exciting discoveries! For instance, if the control form is a square, the set of points will fill the entire area or appears the form of the formerly acquainted Cantor-cloud, pending whether the ratio factor is greater or lesser than 0.5, respectively. Those interested in the topic more intensively, may discover the website cited as [4], and the [1] downloadable materials with the [chaos\\_game\\_in\\_square.mp4](#) video. An interesting scientific application is the so-called Chaos Game Representation (CGR), discussed shortly in Chapter 4.

As a little detour, an interesting logical game is discussed here based on the chaos game covered above (Figure 24). The basic playing field of the game is a grey Sierpinski-triangle with a given resolution. The initial state contains a yellow triangle and a little black dot in the right bottom corner. The task is to move the black dot into the very inside of the yellow triangle (perimeter points do not apply) according to the following rules. The black dot can be moved towards the red, blue and green circles on the three corner points of the figure. From the momentary position the point is moved towards the selected corner, into the bisecting point of the connecting section (as known from the chaos game). The goal is to reach the target field by finding the optimum strategy and the minimum number of steps. Of course, both the yellow target triangle and the resolution of the Sierpinski-triangle can be subject to change (as resolution rises, the target triangle shrinks in size and the task will become more and more difficult). Sierpinski playing fields can be printed out in A4 size with different resolutions from the document entitled [chaos\\_game\\_table.doc](#) found in the downloadable materials [1]. Mark the point to be moved on the table using an X, select the target field and identify the appropriate bisecting points using a ruler over the steps and follow its movements. (Computerised and mobile application implementation of the chaos logical game is anticipated in the near future). Can you get the black point to the target field? Using the resolution on Figure 24 the minimum number of steps is 5. In how many steps did you manage? Can you develop a strategy for all times?

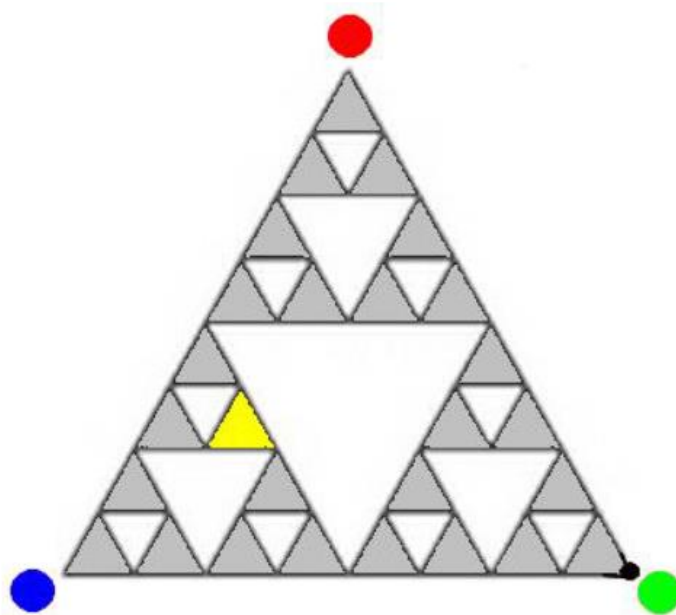


Figure 24: Chaos logic game

### 3.2. Generating landscapes

Generating realistic landscapes is a good example for the application of random iteration algorithms (for movies, video games). A fractal landscape is a surface, which is generated by a stochastic algorithm, designed to create results showing fractal behaviour, thus invoking the illusion

of a natural relief. In other words, a not a predetermined fractal surface is obtained at the end of the operation, much rather a random form carrying fractal like properties. Steps of the most frequently used so called diamond square algorithm is seen on Figure 25.

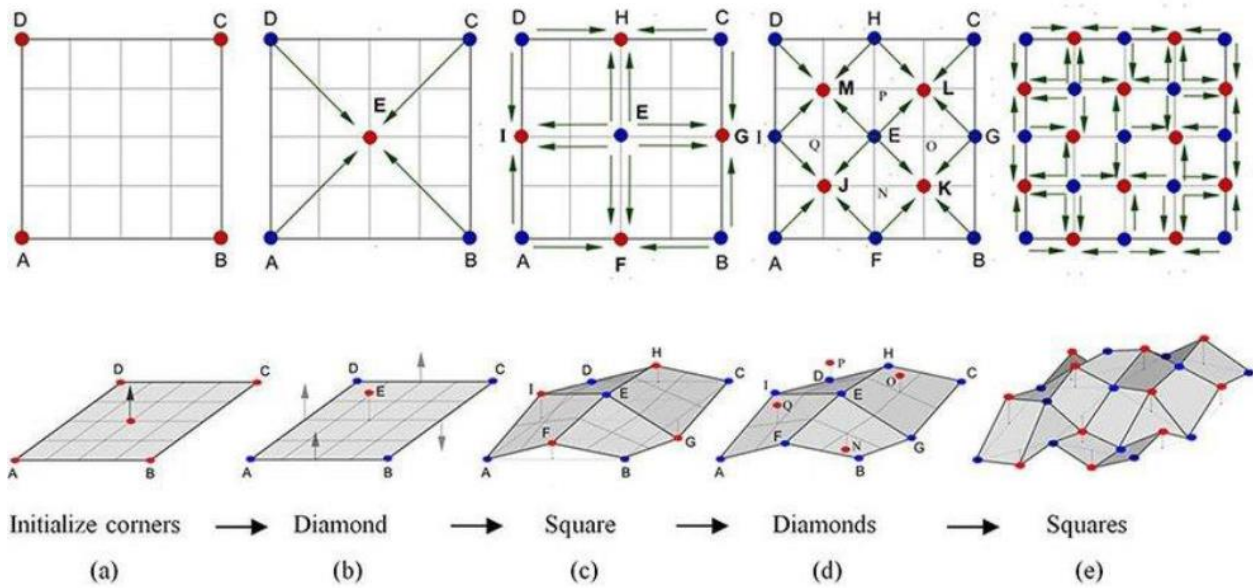


Figure 25: Some steps of the diamond square algorithm

(Source: [https://www.researchgate.net/figure/a-The-method-DS-Algorithm-that-generates-fractal-terrain-Red-dots-represent-the-new\\_fig2\\_320554782](https://www.researchgate.net/figure/a-The-method-DS-Algorithm-that-generates-fractal-terrain-Red-dots-represent-the-new_fig2_320554782))

Figure 26 illustrates the generation of a computer aided fractal landscape.

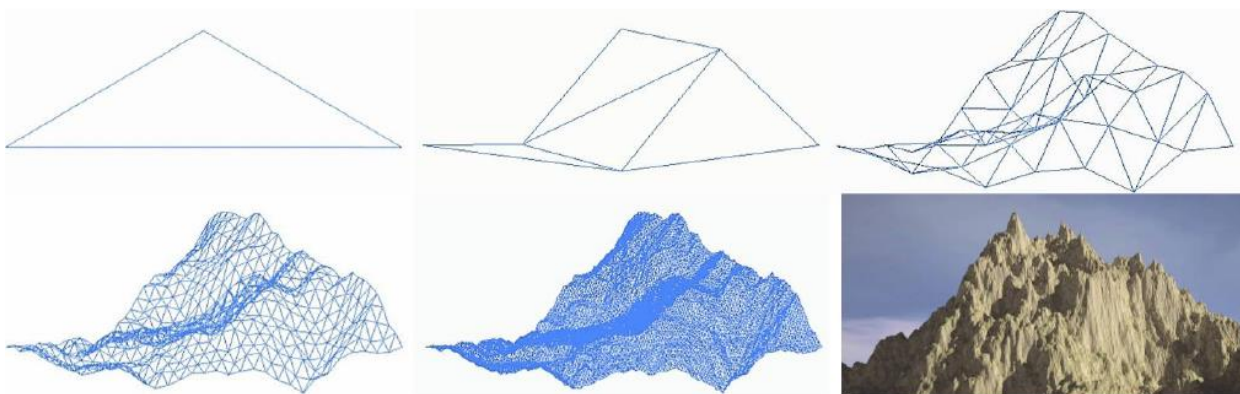


Figure 26: First steps and final result of stochastic fractal landscape generation

In this chapter, thus, it is shown that stochastic fractals grasp the geometry of the material world even more deeply and generally.

#### 4. Fractals in science

Since their discovery, fractals are used in more and more field, ranging from heart beat characterisation through investigating equilibrium, cancer research, stock exchange trends, meteorology, computer graphics, up to digital image processing. In this last chapter of the study a few examples are flashed to emphasise the significance of fractals in the world of science and technology.

The mathematician Richard Taylor analysed the fractal properties of the paintings made by Jackson Pollock using scientific methodology [5]. Such points of interest were surfaced for instance in relation to the paintings of Pollock, that their fractal dimensions, that is their irregularity were increased with the age of the artist, as if the artist had been able to develop his fractal creating capacities. The Pollock-Krasner foundation established by the widow of the early deceased renowned artist handed over six paintings to the research team led by Taylor, to examine the authenticity of the paintings based on their results obtained so far. The mathematical analysis pronounced the paintings to be most probably mere imitations, forgery, with a very slight chance only



that indeed Pollock was the author. An exciting and so far quite efficient new scientific approach, **Chaos Game Representation** is based on the chaos game discussed in the previous chapter, illustrated here with the example of the DNA structure (Figure 27). Information on the architecture of the living matter is found in the base sequence of DNA, i.e. the order of the four bases (**A**dentine, **G**uanine, **C**ytosine and **T**hymine) carries the information. In this case, therefore, the initial controlling form is a square with **A, G, C** and **T** letters (corresponding to the bases, with the algorithm known.

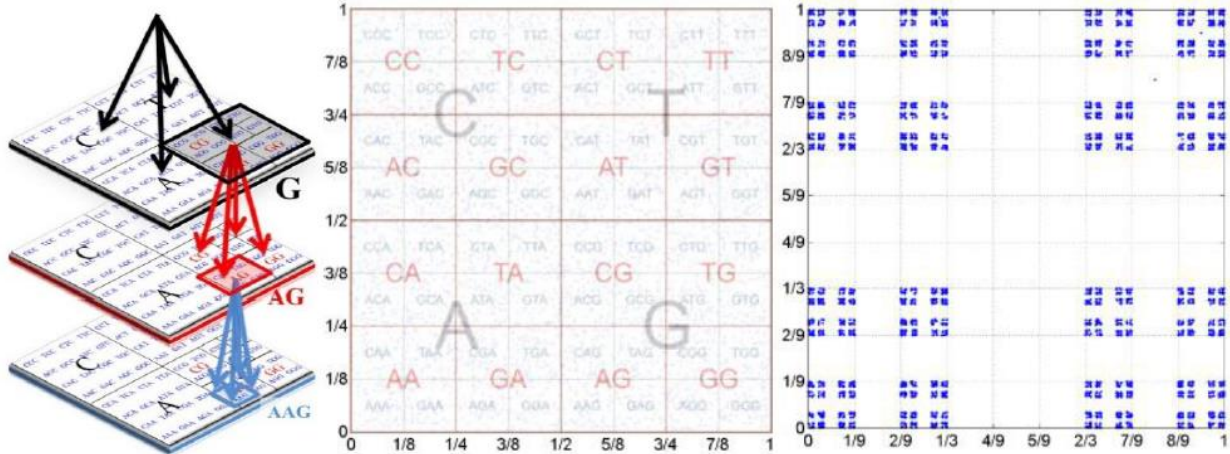


Figure 27: Representation of the amino acid structure of proteins in a chaos game (CGR)  
 (Source: [https://www.researchgate.net/figure/CGR-representation-Chaos-Game-Representation-CGR-map-of-random-DNA-sequences-of-length\\_fig1\\_224886995](https://www.researchgate.net/figure/CGR-representation-Chaos-Game-Representation-CGR-map-of-random-DNA-sequences-of-length_fig1_224886995))

Detailed discussing is far beyond the scope of this study, the description found at the link [6] is recommended for more information, merely its use is mentioned here:

- characterisation and classification of species,
- identification and comparative analysis of organisms
- display and analysis of gene sequences,
- localisation of characteristic parts of gene sequences (such as intron, exon, promoter, transcript factor),
- analysis and identification of proteins.

A number of examples can be found for chaotic processes in medicinal biology just as well. In fact, such processes are fundamental. Frequently the problem is just the discontinuation of the chaotic behaviour and in such cases medicine is charged with the task of restoring chaos. In epilepsy, for instance, the normal chaotic functions of the brain must be put to place. Our body, by the way, is stuffed with fractals. Due to coercion by evolution, living organisms developed energy and space saving solutions. Maybe the most typical such development is the presence of space filling fractals. As seen above, average human lungs extend to a size of a tennis court, yet fit in the chest.

Fractals are infinitely complex, complicated conformations, which can be described sometimes using pretty simple rules. This dichotomy allows compaction of data sets such as images. The encryption of information is one of the most important field of application for fractals and chaos theory. The video entitled [fractal\\_encrypt.mp4](#) found among the downloadable materials [1] provides a little insight into the essence of the procedure.

Note that according to the latest research findings on the large scale architecture of the Universe the distribution of galaxy clusters also shows the nature of a fractal.

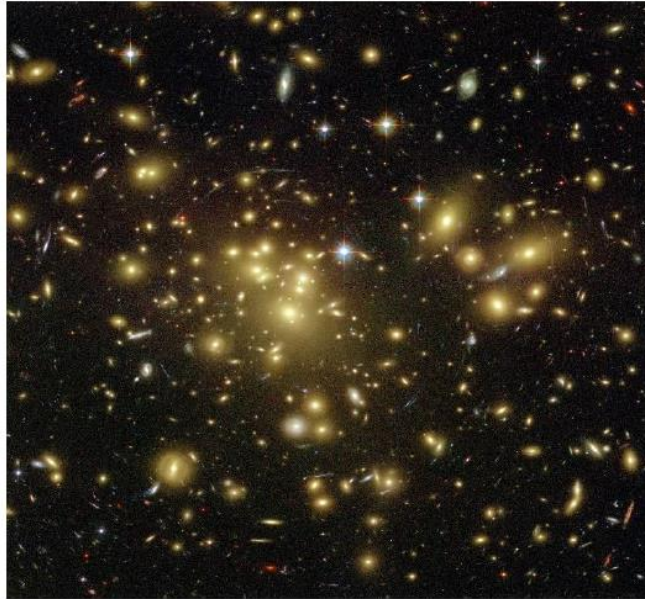


Figure 28: Fractal nature of the large scale structure of the Universe  
(Source: <https://www.dhushara.com/funiv/FractalUniverse.htm>)

Finally, let's take *chaos theory*, the most important and significant area where fractals appear (and one of the research topics of the authors of this paper). Chaos theory is one of the most significant achievement of science in the second part of the 20<sup>th</sup> century. Although they can be described using simple (low degrees of freedom), unequivocal (deterministic) principles (in fact, by a couple of non-linear dynamic equations) chaotic systems move with the following basic properties:

- they show irregular (non-periodical, complicated) behaviour,
- due to the rapid growth of the errors in the initial state and hence, extreme sensitivity to the initial conditions, the behaviour of the system practically is unpredictable for a longer term (thus only a probability description can be provided),
- long term behaviour in the phase space completely defining changes over time is characterised by a special geometric structure, or order (for instance, attractors in the phase space show fractal geometry).

Due to the subject matter of this paper the last aspect is the most important. Physics investigate changes of the properties of the most diverse systems over time, and phase space is a convenient tool to describe such changes. Phase space is an  $n$  dimensions abstract space defined by the number of variables  $x = \{x_1, x_2, \dots, x_n\}$  minimally necessary for unequivocal definition of the state of a dynamic system in any one moment. System state is represented by a point of the phase space in any one moment, and the curve covered by the phase point over time in the development of the system is the *trajectory*. The construction of the phase space is illustrated by the run of the file entitled [phase space.html](#) found among the downloadable materials [1]. The attracting set of the phase space, towards which the trajectories are heading on the long run is called *attractor*, and the attractor of chaotic systems is of fractal nature. Fractal geometry associated with system types and chaotic properties is illustrated on Figure 29.

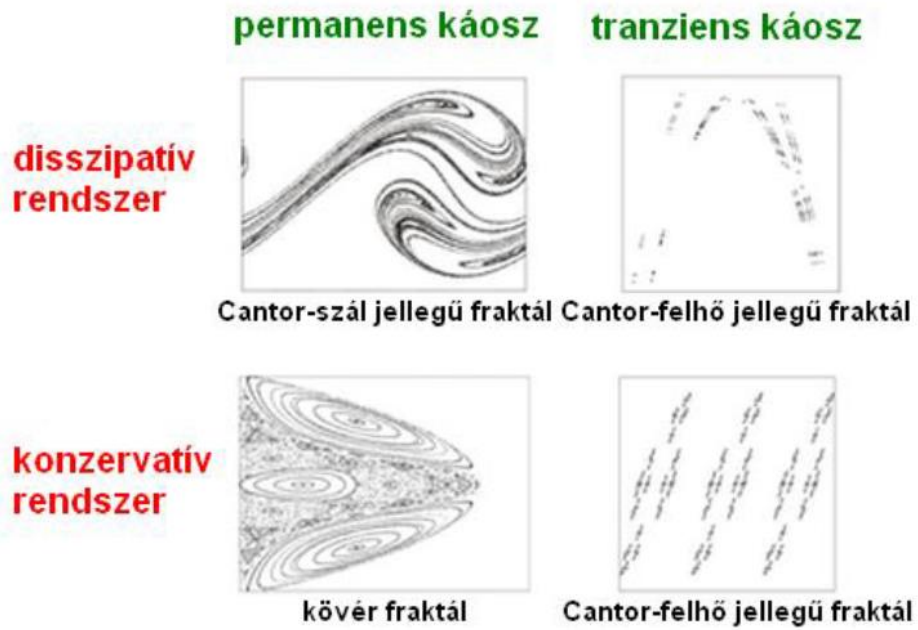


Figure 29: Chaos and fractal geometries

As an example, the self-similar Cantor-thread type chaotic attractor of the frictional non-linear incited oscillation (the so-called *Duffing-oscillator*) can be illustrated on Figure 30 by serially zooming in onto the angular diversion-angular velocity phase plane.

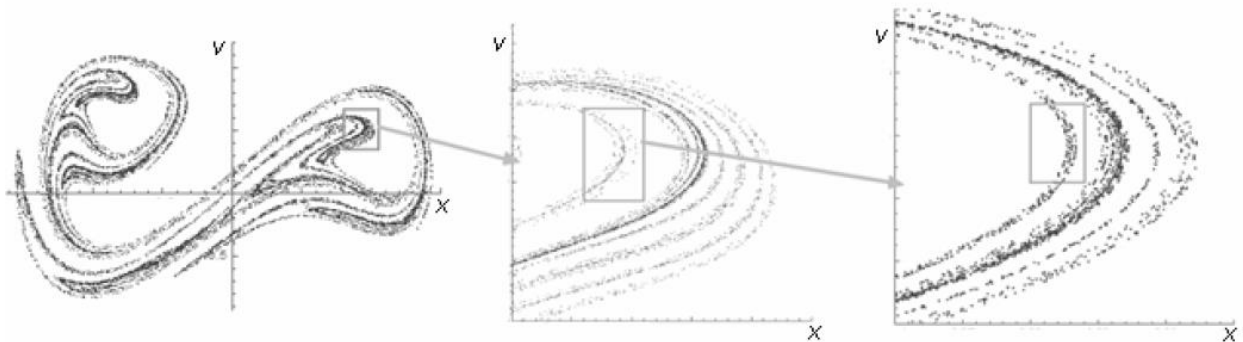


Figure 30: Cantor-thread fractal geometry of the chaotic attractor in the Duffing-oscillator (self-similarity)



One of the simplest, most interesting, commercially available device showing chaotic behaviour is the so called magnet pendulum; Small magnet discs of different colours are placed on the supporting plate under the thread pendulum made of a magnetic body which attract (or, alternatively, in another type, distract) the magnetic body on the end of the pendulum. The pendulum, when started from the diverted position, shows a so called transient chaos, i.e. moves chaotically for a finite period of time, finally stopping above one of the magnet discs due to friction. The attraction range can be mapped when starting the pendulum from different starting points and indicating the initial position with the colour of the magnetic disc above which the pendulum is finally stopped. The map obtained this way has a fractal geometry, and it is recommended to view the video [magnet\\_pendulum.mp4](#) of the downloadable [1] materials.

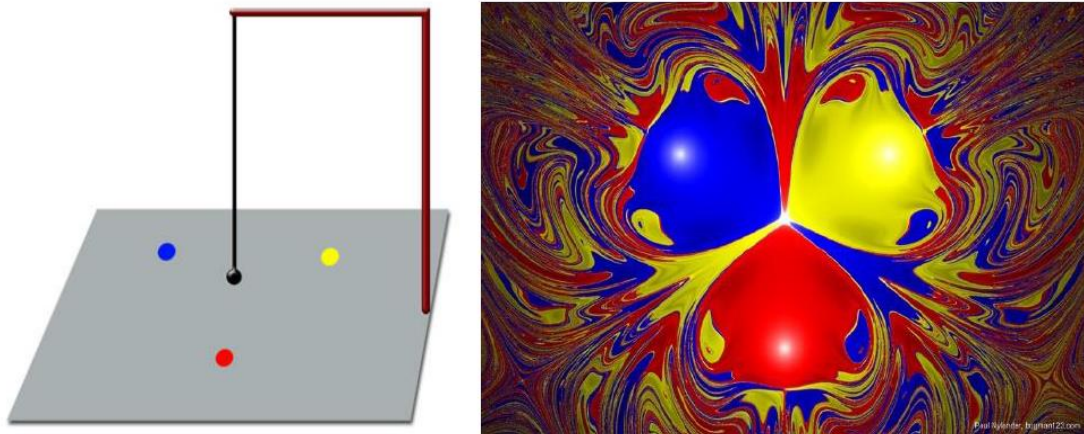


Figure 31: The magnetic pendulum and the map of the fractal geometric attraction range

## References

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