Find the equations of motion in various ways! - From Newton's second law to Lagrangians

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Abstract One of the most important questions of physics teaching is to introduce students to the creative way of problem-solving and to help them from novice to expert type thinking. The development of this capability can be effectively facilitated by producing various solutions to the same problem. This paper shows in a didactic way how one can build the equations of motions of increasingly intricate systems by the use of various theoretical models. Starting with the simple spring pendulum step by step the damped and forced spring pendulum is reached. We are using more ways to find the equations of motion (Newton's second law in inertial and noninertial systems, Lagrange method).

1. Introduction

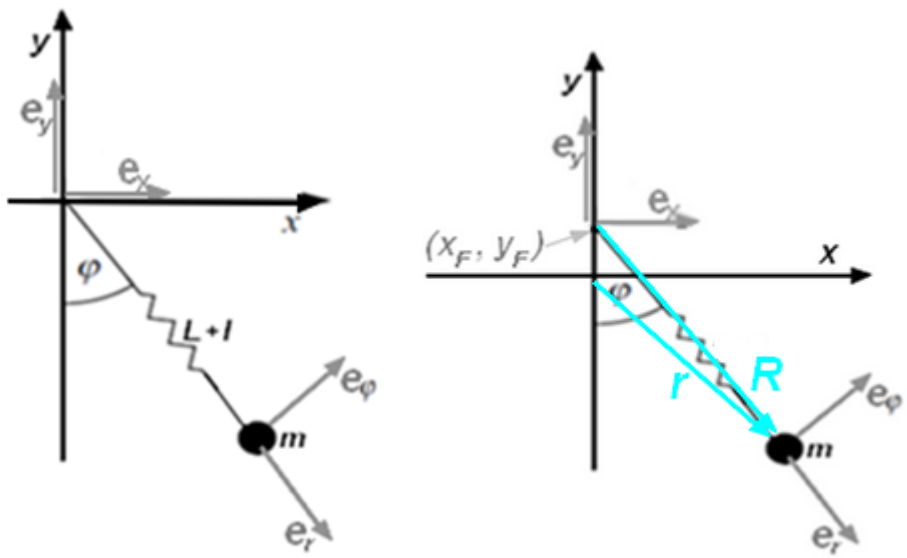
Problem-solving is a core question of physics teaching. There is a huge amount of literature, which discusses this question. Recent reviews have given a detailed picture of the field. [1,2]. Teachers should approach the question of problem-solving with different tactics than experts in physics make. For teachers, the teaching of problem-solving is at least such an important goal than to obtain a good solution which is generally the only aim of an expert. [3] Teachers should think in a flexible way when they instruct a class in problem-solving and they should follow and understand quickly the different trains of thought of the students. It means that it is very advantageous if teachers are able to suggest various solutions for the same problem. According to our experience, these didactic skills of future teachers can be improved at university training if various solutions for the same problems are presented. Besides this, we propose the application of the method of "gradually more difficult problems". That means that series of problems are constructed where every problem originates from the previous one by a slight modification.

The description of the motion of bodies is one of the most important parts of physics, so we have taken problems from this field to illustrate the variety of possible solutions. It seems to us that the solution of a proper row of problems that contain step by step more difficult problems can facilitate students to reach a deeper understanding of the mechanical concepts and also to improve the flexibility of their thinking. Earlier this method was applied successfully in secondary school for teaching chaotic motion with studying the motion of Duffing oscillator as an example [4]. Although in this previous paper the emphasis was put on studying the chaotic properties of the solution, the mathematical model of the

physical system was built up by adding forces step by step to the model as we suggest in the present paper. In this paper, a series of problems based on a spring pendulum is presented and will be solved by different methods which leads from secondary school physics to the use of Lagrangians. The spring pendulum provides possibilities to apply different methods and by choosing proper parameters its solution exhibits a wide range of trajectories from regular to chaotic. In this paper, we focus on the construction and the analysis of the mathematical model of the systems investigated, and the solution, which is at least so interesting as the model construction, will be only flashed. The detailed description of the solution of the equations of motion will be published in a further publication.
Of course, the problem solving connected to university courses cannot be a self-important one it should target to enlighten or deepen some important law or concept of physics. In this paper, it will be also shown that how we can help the deeper understanding of the force concept which plays a central role in Newtonian mechanics.

2. The problems

Our starting problem is the well-known spring pendulum [5]:
P1. A small size body of mass m is suspended on a weightless spring with a spring constant *D*. The length of the unstretched spring is *L*. Displace the pendulum from its equilibrium and release it! Describe the motion! A great advantage of this problem is that using proper parameters both regular and irregular behaviour of the pendulum can be demonstrated experimentally very simple way, which generally arouses the interest of the students.



a. problems **P1** and **P2** **b**. problem **P3**
Figure 1. .Basis vectors of Descartes system and polar coordinate system

This problem can be easily made more complex and more difficult:
P2. Describe the motion if the pendulum is damped by a viscous force which proportional and opposite with the velocity of the pendulum.
P3. Describe the motion of the damped pendulum if its suspension point is forced vertically according to a harmonic function $y_F(t) = A \cdot \cos\left(\frac{2\pi}{T} t\right) = A \cdot \cos(\Omega t)$, where A is the amplitude of the forced motion of the suspension point, and Ω is the angular frequency.
Everyday realization of the forced pendulum is for example the well-known yo-yo toy, where the suspension point of a spring pendulum (it is a ball fixed on a rubber thread) is pulled up and down.

Some real motions are shown on downloadable videos from our page on the following links: the spring pendulum (**P2**) *http://csodafizika.hu/springpendulum/spring_pendulum.mp4* and the forced pendulum (**P3**) *http://csodafizika.hu/springpendulum/driven_spring_pendulum.mp4*.

3. Overview of the forces

The problems can be solved by Newton's law and the Lagrange equations. Nevertheless, the physics of the problems can be more easily explored on the basis of the Newtonian method. Therefore first the equations of motion will be written in an inertial reference system in the simplest possible way using vector notations, but without getting involved in the details of the calculations.
Since the forces acting are the spring force and the gravitation force, we get from the Newton equation $m\mathbf{a} = \sum \mathbf{F}$, the equation of motion for the pendulum **P1.**:

$$m\mathbf{a} = -Dl\mathbf{e}_r - mg\mathbf{e}_y$$

where the notations of Fig. 1 are used and l is the elongation of the spring, D is the spring constant. The pendulum bob fixed to the end of the spring is at **r**(*x,y*), the length of the spring is $L + l = \sqrt{x^2 + y^2}$, and the force arising in it is $F_s = -D\left(\sqrt{x^2 + y^2} - L\right)$

In **P2** this equation should be modified by adding a viscous damping force which is proportional and opposite to the relative $\mathbf{v}_I$ velocity of the pendulum bob through the resting air. So the damping force law is: $\mathbf{F}_D = -c\mathbf{v}_I$, where c is the damping constant.
According to this the Newton equation of motion is:

$$m\mathbf{a} = -Dl\mathbf{e}_r - mg\mathbf{e}_y - c\mathbf{v}_I$$

In **P3** a periodic forcing is applied at the suspending point of the pendulum. It is a kinematic constraint, which control the position of the line of the pendulum spring as it is shown in Fig 1.b. $\left(\mathbf{R} = R\mathbf{e}_R = \mathbf{r} - y_F\mathbf{e}_y\right)$ and due to this

$$L + l = \left|\mathbf{R} - y_F\mathbf{e}_y\right| = \sqrt{x^2 + \left(y - y_F\right)^2}$$

Consequently, the spring force is $F_S = -Dl\mathbf{e}_R$ and the equation of motion will be:

$$m\mathbf{a} = -Dl\mathbf{e}_R - mg\mathbf{e}_y - c\mathbf{v}_I$$

Table 1. Characterization of the problems

	P1	**P2**	**P3**
equation of motion	$m\mathbf{a} = -Dl\mathbf{e}_r - mg\mathbf{e}_y$	$m\mathbf{a} = -Dl\mathbf{e}_r - mg\mathbf{e}_y - c\mathbf{v}_I$	$m\mathbf{a} = -Dl\mathbf{e}_R - mg\mathbf{e}_y - c\mathbf{v}_I$
type of system	conservative	dissipative	excited dissipative
chaotic behavior	regular and chaotic regions on Poincare map	transient chaos	permanent chaos or transient chaos with fractal basin

Although these equations describe well the motion of the systems, it is worth choosing coordinates that fit better to the systems so give a clearer picture about them. The motion of the spring pendulum

can be imagined as a superposition of the swinging motion of a pendulum and the oscillation of a spring. The swinging motion can be described with the angular displacement φ of the spring line, while the oscillation can be described with the elongation l of the spring.
These quantities characterize the motion much relevant way since they independently demonstrate the two motion components. Evidently, we can understand better the motion if we choose a coordinate system that is explanatory in itself. We can find a coordinate system in which the two movements are characterized with the above coordinates, φ and l; it is the polar coordinate system. A further good coordinate system may be that, which rotates together with the pendulum. In this system, the vibration of the spring is the only motion. It is worth comparing the description of the spring pendulum in these systems.

In the following **P1** and **P3** will be solved in various coordinate systems which fit to the physics of the motion. We omit **P2** for the sake of brevity because every essential steps of its solution appear in **P3**. Of course at an introductory university course it is not worthwhile to deviate from the didactically accepted principle of going from the simple to the complex.

4. Solution of P1

4.1. Polar coordinate system

Using the polar coordinate system whose reference point and reference direction are the origin and the –y axis of the previously defined Descartes system, the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are shown in Fig. 1. In this system the position of the bob is $\mathbf{r} = r\mathbf{e}_r$. However, when the position of the bob is changing than the basis vectors of the polar coordinate system are also changing. Therefore $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\varphi}\dot{\mathbf{e}}_\varphi$, so the velocity coordinates of the moving body do not agree with the derivatives of the position coordinates. Similarly, the coordinates of the acceleration do not agree with the derivatives of the velocity coordinate. It can be proved [6] that the components of the acceleration are:

$$\boldsymbol{a}_r = \left(\ddot{r} - r\cdot\dot{\varphi}^2\right)\cdot\mathbf{e}_r$$
$$\boldsymbol{a}_\varphi = \left(r\cdot\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)\cdot\mathbf{e}_\varphi$$

To get the equations of motion the forces have to be expressed by the polar coordinates The gravity force is $-mg\mathbf{e}_y$, and its components in the polar coordinate system are ($-mg\cos\varphi\mathbf{e}_r, mg\sin\varphi\mathbf{e}_\varphi$) while the spring force is simply $-Dl\mathbf{e}_r$. The equations of motion in the polar coordinates are:

$$m\cdot\left(\ddot{r} - r\cdot\dot{\varphi}^2\right) = mg\cdot\cos\varphi - Dl$$

$$m\cdot\left(r\cdot\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right) = -mg\cdot\sin\varphi$$

Having used $r = l + L$ as well as $\dot{r} = \dot{l}$ and $\ddot{r} = \ddot{l}$ as a consequence of L is a constant, the equations of motion will be: Since $\dot{r} = \dot{l}$ and $\ddot{r} = \ddot{l}$ the equations can be written in the form:

$$\left.\begin{aligned}\ddot{l} &= -\frac{D}{m}l + g\cdot\cos\varphi + (L+l)\cdot\dot{\varphi}^2 \\ \ddot{\varphi} &= -\frac{g}{L+l}\cdot\sin\varphi - \frac{2\dot{l}}{L+l}\cdot\dot{\varphi}\end{aligned}\right\}$$

Let us notice the terms $(L+l)\cdot\dot{\varphi}^2$ and $-\frac{2\dot{l}}{L+l}\cdot\dot{\varphi}$ that are at the right hand side of the equations. These are the consequence of the change of the basis vectors therefore they are called metric accelerations. [7].

4.2. Rotating system

In a coordinate system rotating together with the pendulum, the whole motion is restricted to the vibration of the spring. The basis vectors of the polar coordinate system do not change because they rotate together with the pendulum. In this case the coordinates of the velocity and the coordinates of the acceleration agree with the derivatives of the coordinates of the position and the derivatives of the velocity coordinates, respectively. But unfortunately, the Newton's law is valid only if inertial forces are taking into account. The radial acceleration is:

$$\boldsymbol{a}_r = \frac{d^2r}{dt^2}\boldsymbol{e}_r = \ddot{r}\cdot\boldsymbol{e}_r$$

The tangential acceleration is:

$$\boldsymbol{a}_\varphi = r\cdot\frac{d^2\varphi}{dt^2}\cdot\mathbf{e}_\varphi = r\cdot\ddot{\varphi}\cdot\mathbf{e}_\varphi$$

The real forces are the gravity and the spring forces. The gravity force is $-mg\mathbf{e}_y$, and its components in the polar coordinate system are ($-mg\cos\varphi\mathbf{e}_r$, $mg\sin\varphi\mathbf{e}_\varphi$) The spring force is $-Dl\mathbf{e}_r$.

Due to the rotating system inertial forces are also acting; the centrifugal force $(m\cdot r\cdot\dot{\varphi}^2\mathbf{e}_r)$, and the Coriolis force $(-m\cdot 2\dot{r}\cdot\dot{\varphi}\mathbf{e}_\varphi)$ .Consequently the equations of motion are:

$$m\ddot{r} = mg\cos\varphi - Dl + mr\dot{\varphi}^2$$
$$mr\cdot\ddot{\varphi} = -mg\sin\varphi - m2\dot{r}\dot{\varphi}$$

Of course, these equations agree with those obtained in the inertial system. However, their interpretation is rather different!

Since $\dot{r} = \dot{l}$ and $\ddot{r} = \ddot{l}$ the equations can be written in the form:

$$\begin{cases}\ddot{l} = -\frac{D}{m}l + g\cos\varphi + (l+L)\dot{\varphi}^2 \\ \ddot{\varphi} = -\frac{g\cdot\sin\varphi}{l+L} - \frac{2\dot{l}\dot{\varphi}}{l+L}\end{cases}$$

These are the same as those were obtained in the resting polar coordinate system. The two terms expressing the inertial forces formally agree with those originating from the metric accelerations, but their interpretation is far different. (The difference will become very important in the theoretical physics at the introduction of the curvilinear coordinates.)

4.3. The Lagrange description

In mechanics, the time development of every dynamical system can be described by the Lagrangian formalism derived from the least action principle. The time development of the system can be obtained from the Euler-Lagrange equations [8]:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_i} = 0$$

where $\mathcal{L} = T - V$ is the Lagrange function, T and V are the kinetic and the potential energy of the system, respectively. Φ_i denote the general coordinates of the system which are in the case of the spring pendulum l and φ. The Euler Lagrange equations of the spring pendulum are:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial l} - \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{l}} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \varphi} - \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0 \end{cases}$$

The kinetic and potential energies:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \text{ , } V = m \cdot g \cdot y + \frac{1}{2}D \cdot l^2$$

where

$$x(t) = (L + l) \cdot \sin\varphi \text{ , } y(t) = -(L + l) \cdot \cos\varphi$$

After a tiresome but straightforward calculation we got the equations which agree with that obtained in polar coordinates:

$$\begin{cases} \ddot{l} = (l + L)\dot{\varphi}^2 + g\cos\varphi - \dfrac{D}{m}l \\ \ddot{\varphi} = \dfrac{-2\dot{l}\dot{\varphi} - g \cdot \sin\varphi}{l + L} \end{cases}$$

The details of the calculation can be seen at [6]. To solve these differential equations we introduce the velocity $v = \dot{l}$ and the angular velocity $\omega = \dot{\varphi}$. Using these new variables we got the standard form of the equations of motion:

$$\begin{cases} \dot{l} = v \\ \dot{v} = (L + l)\omega^2 + g\cos\varphi - \dfrac{D}{m} \cdot l \\ \dot{\varphi} = \omega \\ \dot{\omega} = \dfrac{-g}{(L + l)}\sin\varphi - \dfrac{2}{(L + l)}v\omega \end{cases}$$

It can be seen that the equations of the motion can be derived by various but equivalent methods. In introductory courses, the use of different methods can reinforce each other. The equations of motion show well the physical background of the terms in the equations, while the derivation of the equations from the Lagrange function is a straightforward procedure that almost automatically leads to the result.

However, the latter does not give guidance to the physics of the process investigated. It is very important to note that there are no forces in the Lagrangian method, which makes the interpretation of the results difficult.

5. Solution of P3

Investigating the physics of the motion in **P3** it can be seen that if the coordinate system is fixed to the suspension point driven periodically then in this system the motion is the same as the motion of a damped pendulum in a resting system.

*5.1. Reducing **P3** to **P***

So it is worth using a coordinate system that moves vertically with the suspension point. In this reference system, the motion of the pendulum agrees with that occurs in the inertial system. Using the polar coordinate system in this accelerating system, the solution can be built upon the solution of the introductory problem. As a first step we should write the equation of motion in the polar coordinate system which is embedded into the accelerating reference system fixed to the suspension point of the spring:

$$m\mathbf{a} = \mathbf{F}_G + \mathbf{F}_S + \mathbf{F}_D + \mathbf{F}_I$$

In this equation the real forces are the gravitational force $\mathbf{F}_G = -mg\mathbf{e}_y$, the spring force $\mathbf{F}_S = -Dl\mathbf{e}_R$, the damping force $\mathbf{F}_D$ and the inertial force $\mathbf{F}_I = -ma_F\mathbf{e}_y = mA\Omega^2\cos(\Omega t)\mathbf{e}_y$, where the vertical unit vector $\mathbf{e}_y = -\cos\varphi\mathbf{e}_r + \sin\varphi\mathbf{e}_\varphi$ and

To find the suitable mathematical form of the damping force is not simple, because the velocity ($\mathbf{v}_I$) with respect to the inertial system should be expressed with the polar coordinates taken in an accelerating coordinate system.

Taking into account that the position vector $\mathbf{r}$ in the inertial system is $\mathbf{r} = R + y_F\mathbf{e}_y$ and $\mathbf{R} = R\mathbf{e}_R$ as well as $\dot{\mathbf{R}} = \dot{R}\mathbf{e}_R + R\dot{\varphi}\mathbf{e}_\varphi$,where the velocity of the bob in the inertial system is: $\dot{\mathbf{r}} = \mathbf{v}_I = \dot{\mathbf{R}} + \dot{y}_F\mathbf{e}_y$. Substituting $\dot{\mathbf{R}}$ and $\mathbf{e}_y$, we get

$$\mathbf{v}_I = \dot{R}\mathbf{e}_R + R\dot{\varphi}\mathbf{e}_\varphi + \dot{y}_F\mathbf{e}_y = \dot{R}\mathbf{e}_R + R\dot{\varphi}\mathbf{e}_\varphi + A\Omega\sin(\Omega t)\cos\varphi\mathbf{e}_R - A\Omega\sin(\Omega t)\sin\varphi\mathbf{e}_\varphi$$

Using $R = L + l$, the damping force can be written as:

$$\mathbf{F}_D = -c\dot{l}\mathbf{e}_R - c\left(l + L\right)\dot{\varphi}\mathbf{e}_\varphi - A\Omega\sin(\Omega t)\cos\varphi\mathbf{e}_R + A\Omega\sin(\Omega t)\sin\varphi\mathbf{e}_\varphi$$

Applying the formula concerning the accelerations in polar coordinates:

$$\boldsymbol{a}_R = \left(\ddot{R} - R\dot{\varphi}^2\right)\mathbf{e}_R = \left[\ddot{l} - \left(L + l\right)\dot{\varphi}^2\right]\mathbf{e}_R \quad \text{and} \quad \boldsymbol{a}_\varphi = \left(l\cdot\ddot{\varphi} + 2\dot{l}\dot{\varphi}\right)\cdot\mathbf{e}_\varphi$$

The components of the newtons laws will be:

$$\begin{cases} \ddot{l} = \left(l + L\right)\dot{\varphi}^2 + g\cos\varphi - \dfrac{D}{m}l - A\Omega^2\cos\theta\cos\varphi - \dfrac{c}{m}\dot{l} - \dfrac{c}{m}A\Omega\sin\theta\cos\varphi \\ \ddot{\varphi} = \dfrac{-2\dot{l}\dot{\varphi} - g\cdot\sin\varphi + A\Omega^2\cos\theta\sin\varphi + \dfrac{c}{m}A\Omega\sin\theta\sin\varphi}{l + L} - \dfrac{c}{m}\dot{\varphi} \end{cases}$$

Rearranging the system to a system of first-order differential equations we got the standard form of the equations of motion:

$$
\begin{cases}
\dot{l} = v \\
\dot{v} = (l+L)\omega^2 + g\cos\varphi - \dfrac{D}{m}l - A\Omega^2\cos\theta\cos\varphi - \dfrac{c}{m}v - \dfrac{c}{m}A\Omega\sin\theta\cos\varphi \\
\dot{\varphi} = \omega \\
\dot{\omega} = \dfrac{-2v\omega - g\cdot\sin\varphi + A\Omega^2\cos\theta\sin\varphi + \dfrac{c}{m}A\Omega\sin\theta\sin\varphi}{l+L} - \dfrac{c}{m}\omega \\
\dot{\theta} = \dfrac{2\pi}{T}
\end{cases}
$$

We should notice that as in the case of P1, here, a rotating coordinate system for the description of the motion can be also relevantly used. It might be very instructive because the rotating system ought to be embedded into the accelerated system of the driving force. However, in the lack of space, it is left to the reader.

5.2. Solution with the Lagrange formalism

The original form of Lagrange equations are valid for systems where only conservative forces are acting. It means that potential energy can be associated with forces. So dissipative forces as the frictional and viscous drag cannot be treated by Lagrange formalism. However, the Lagrange equations can be completed by non-conservative or generalized forces Q_i which are suitable to take into account the friction [8]:

$$\frac{\partial\mathcal{L}}{\partial\Phi_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_i} = Q_i$$

In the case of the viscously damped pendulum the generalized force Q_i can be derived from the Rayleigh dissipation function $D_R = \frac{c}{2}|v|^2$ as $Q_i = \frac{\partial D_R}{\partial\dot{\Phi}_i}$.(Landau) The Rayleigh dissipation function comes from the thorough investigation of the forces acting in the system. The modified Lagrange equation is:

$$\frac{\partial\mathcal{L}}{\partial\Phi_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_i} = \frac{\partial D_R}{\partial\dot{\Phi}_i}$$

In our case:

$$
\begin{cases}
\dfrac{\partial\mathcal{L}}{\partial l} - \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot{l}} = \dfrac{\partial D_R}{\partial\dot{l}} \\
\dfrac{\partial\mathcal{L}}{\partial\varphi} - \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dfrac{\partial D_R}{\partial\dot{\varphi}}
\end{cases}
$$

where:

$$\dot{x} = \dot{l}\cdot\sin\varphi + (L+l)\cdot\cos\varphi\cdot\dot{\varphi} \;\;,\;\; \dot{y} = \dot{y}_F - \dot{l}\cdot\cos\varphi + (L+l)\cdot\sin\varphi\cdot\dot{\varphi}$$

The Lagrange function:

$$\mathcal{L} = T - V = \frac{1}{2} m \cdot \dot{y}_F^2 + \frac{1}{2} m \cdot \dot{l}^2 + \frac{1}{2} m \cdot (L+l)^2 \cdot \dot{\varphi}^2 + m \cdot \dot{y}_F \cdot (L+l) \cdot \sin\varphi \cdot \dot{\varphi} -$$

$$- m \cdot \dot{y}_F \cdot \dot{l} \cdot \cos\varphi - -m \cdot g \cdot y_F + m \cdot g \cdot (L+l) \cdot \cos\varphi - \frac{1}{2} D \cdot l^2$$

Substituting this function into the Lagrange equations, after a tiresome but straightforward calculation we got the equations of motion in the form of second order differential equations (the details can be find at [6]):

$$\begin{cases} \ddot{l} = \ddot{y}_F \cdot \cos\varphi + (L+l) \cdot \dot{\varphi}^2 + g \cdot \cos\varphi - \frac{D}{m} \cdot l - \frac{c}{m} \cdot \dot{l} + \frac{c}{m} \cdot \dot{y}_F \cdot \cos\varphi \\ \ddot{\varphi} = -\frac{g + \ddot{y}_F}{(L+l)} \cdot \sin\varphi - \frac{2}{(L+l)} \cdot \dot{l} \cdot \dot{\varphi} - \frac{c}{m} \cdot \dot{\varphi} - \frac{c \cdot \dot{y}_F}{m(L+l)} \cdot \sin\varphi \end{cases}$$

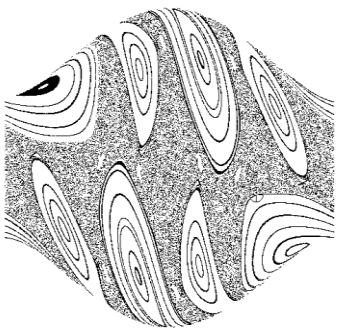
Substituting $\ddot{y}_F = -A \cdot \Omega^2 \cos\theta$ and $\dot{y}_F = -A \cdot \Omega \sin\theta$ as well as introducing $\dot{l} = v$ and $\dot{\varphi} = \omega$ the equations can be rearranged into a system of first order differential equations.

$$\begin{cases} \dot{l} = v \\ \dot{v} = -A\Omega^2 \cos\theta \cos\varphi + (L+l)\omega^2 + g \cos\varphi - \frac{D}{m} \cdot l - \frac{c}{m} \cdot v - \frac{c}{m} A\Omega \sin\theta \cos\varphi \\ \dot{\varphi} = \omega \\ \dot{\omega} = \frac{A\Omega^2 \cos\theta - g}{(L+l)} \sin\varphi - \frac{2}{(L+l)} v\omega - \frac{c}{m} \omega + \frac{cA\Omega \sin\theta}{m(L+l)} \sin\varphi \\ \dot{\theta} = \frac{2\pi}{T} \end{cases}$$

It can be said again that the Newtonian interpretation gives a clear explanation of the terms on the right hand side of equations, while the Lagrange method is more mechanistic and leads to the result in a simple but tiresome way. The Newtonian interpretation of the equation enlightens the physical meaning of the Rayleigh potential too. These systems agree with that obtained on the Newtonian way.

6. A glimpse on the chaos

Finally, we present an exhibition about the beauty of the chaotic trajectories. This pictures demonstrate that the problems chosen are very representative examples of systems with low degree of freedom.



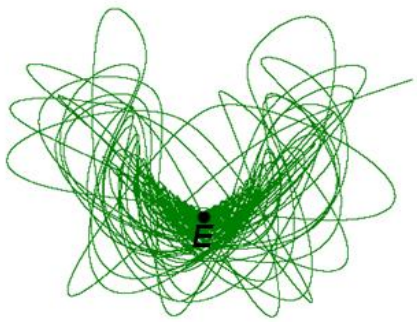


Figure 2.a. Poincare map of it in **P1** **Figure 2.b.** A trajectory of transient chaos shown by the solution of **P2**.the trajectory move into stable equilibrium ***E***)

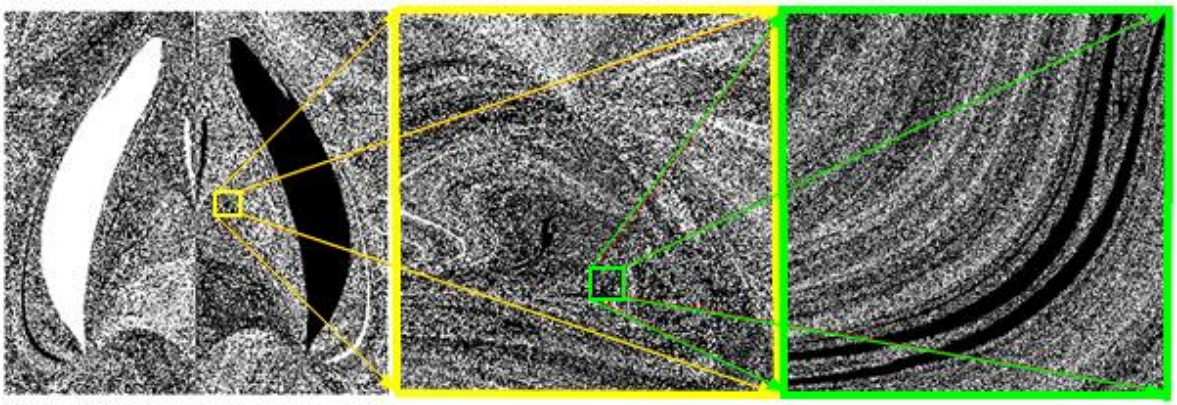


Figure 3. Cantor fiber fractal structure in boundary of attraction basins of **P3**.

7. Conclusion

The main goals of this presentation are to give an example of the use of a step by step more intricate row of problems, which can be used to introduce students to a creative way of problem-solving. We have discussed the possibility of various solutions to a series of mechanical problems and have shown how the mathematical models of physical systems can be created by applying different physical principles.

According to our experience students often struggle with difficulties in reconciling the interpretation of the results obtained by different methods. Especially, the interpretation of the terms appearing in curvilinear coordinate systems, and in the Lagrange method causes great difficulty for them.

Finally, it is worth mentioning that the solution of the differential equations obtained for both P1. and P3 are generally chaotic so it in case P1 shows the conservative and in the case of P3 the dissipative chaos. Furthermore, the solution of P3 exhibits the so called transient chaos which is a very exciting and relatively new field of chaotic motion [9].

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