Studying Non-linear and Chaotic Phenomena in High School

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Abstract

The purpose of this article is to give an outline of the research methods my pupils usually study non-linear phenomena with. Talented and motivated kids with good computer skills apply computer programs to measure and describe complicated dynamics. By the example of a pendulum experiment, I will show that they are competent not only in studying the simple pendulum movement, but the driven dynamics as well. We will present (in detail) the experimental investigation of the driven pendulum, and its numerical analysis. The assemblage of the set-ups, the measurements, the analyses and the computer simulations were made by 14 to 18–year-old members of the 'science-workshop' at our school.

1. Introduction

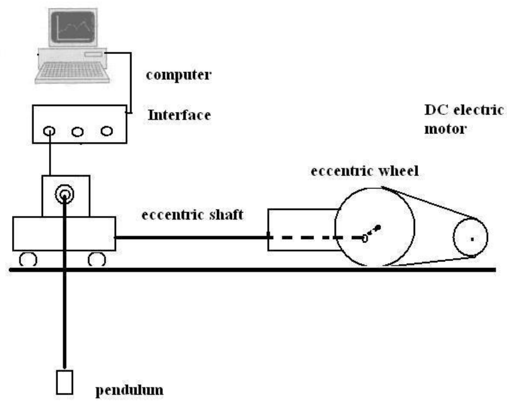
Our workshop was established in 1999 with the aim to go beyond high school physics. We have 8-10 members every year, students 14-18 years of age. In the extracurricular workshop we have 2 sessions a week (and some extra hours). We measure mechanical and electronic systems via computers and simulate them numerically. The young students (14-15 year-old ones) collect data by computer or build the set-ups. The older students (16-18 year-old ones) develop programs and set-ups, evaluate the measured data. We have investigated non-linear systems, e.g., the behaviour of a real transformer and a double physical pendulum.

Why do we study non-linear phenomena? Because here we find surprisingly complex motions of single systems. Such systems cannot be predicted a long time ahead. We face unpredictability and chaos. The example chosen for this article is a horizontally driven damped pendulum. This problem is relatively easy to understand, an experimental set-up is easy to build, and, moreover, it is easy to simulate the dynamics numerically.

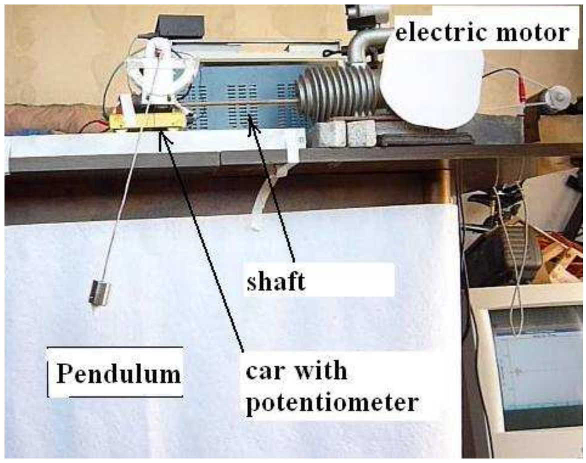
The paper is organized as follows. In the next Section, we describe the experiment and present the main results. Section 3 is devoted to the numerical simulation of an idealized version of the problem. In Section 4 we present our conclusions.

2. Experiments with the driven pendulum

In Fig.1a. we can see the schematic arrangement of the set-up. The pendulum is hung up on an axis of a potentiometer fixed to a car on a track. The voltage of the potentiometer leads to a computer via an interface. An eccentric shaft moves the car with the pendulum and thus provides a periodic driving. A photograph of our set-up is shown in Fig.1b. The car is pushed by means of a transformed model engine.



(a)

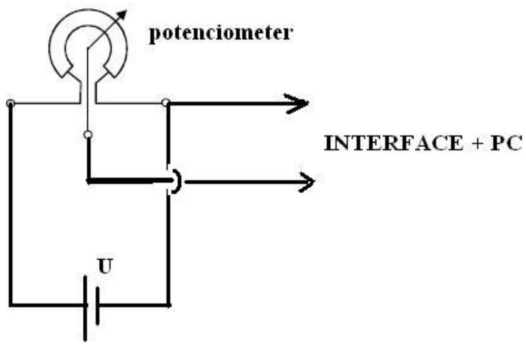


(b)

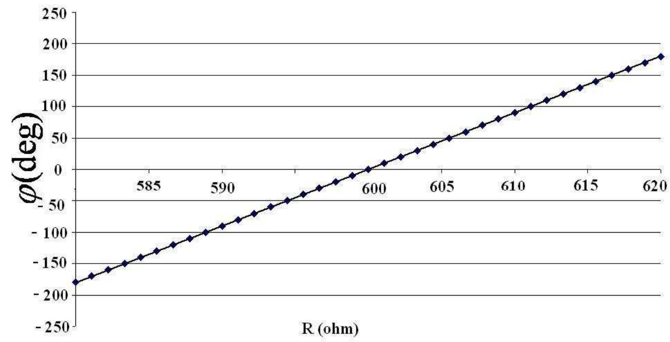
Figure.1: Schematic arrangement (a) and photo (b) of our driven pendulum set-up

We applied a potentiometer of type Helipot, which is a three-terminal resistor with a sliding contact that forms an adjustable voltage divider [1]. Its resistance varies between 3 and 700 ohms. The voltage measured between the middle and the right leads is proportional to the resistance and, in this way, to the deflection angle φ of the pendulum (Fig.2a.). We used U= 8 volts. The potentiometer was calibrated at two positions of the pendulum. One of them was the vertical equilibrium and the other an angle of 20 degrees.

If the resistance R is smaller than 600 ohms, the angle is negative. 10 over-turnings in the positive direction yield 700 ohms. The experimentally obtained relation $R= 0.11\varphi + 600$ between angle φ (measured in degrees) and resistance R (measured in ohm) is shown in Fig.2b. After calibrating, software (Data Monitor for Windows) can collect the appropriate angles. It is able to analyse the collected data by the aid of statistical, mathematical functions. So we can fit some function to the measured data, smooth, integrate, derivate these functions to time. The software is able to graph data with time or according to other data. The students learn to use this software during the extracurricular lessons.



(a)



(b)

Figure.2: Circuit of the potentiometer (a) and the measured linear relation between angle and resistance (b)

As a warm-up, we investigated the damped, non-driven pendulum on a fixed car. We pulled the pendulum approximately φ_0 =46 degrees of its equilibrium and let it move without any initial angular velocity: $\omega_0 = 0$. The swing calmed down because of the friction at its axes and the drag. We can see this in Fig.3.a in an angle vs. time graph.

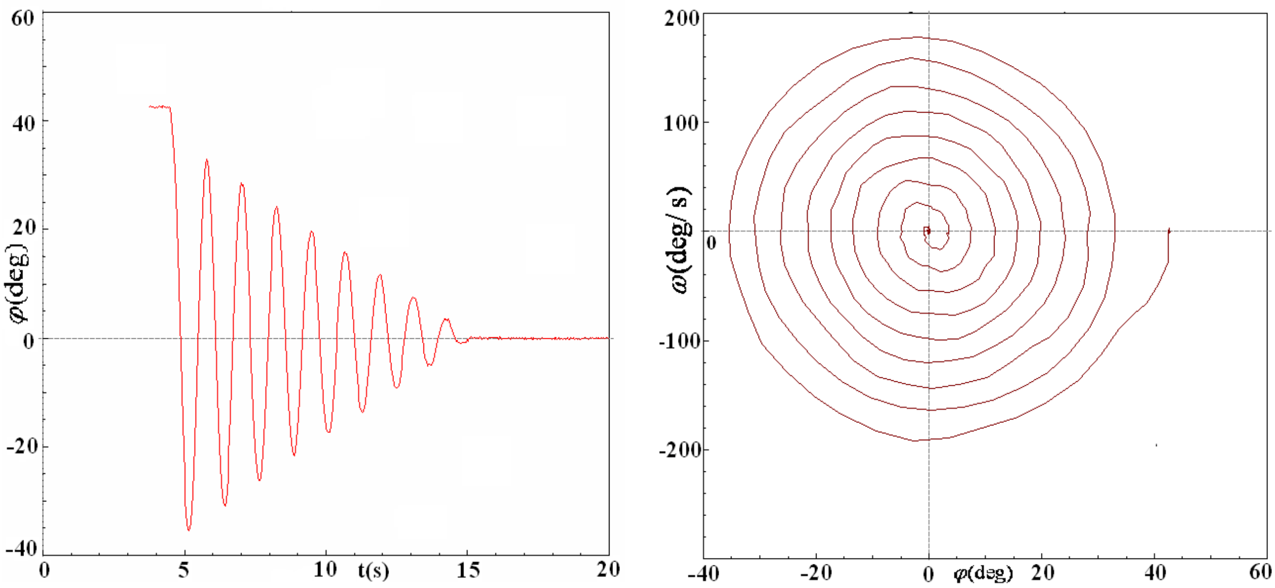


Figure.3: Measured angle vs. time for a damped pendulum with φ_0 = 46 (deg) , ω_0 = 0 (deg/s) (a) and the phase space trajectory of the damped pendulum (b)

Another useful diagram to analyse the motion is the phase space one (Fig.3.b). It shows the relation between angular velocity and angle. We get the angular velocity by differentiating the measured angles numerically with respect to time. We can see a spiral which leads to a point, the origin. This corresponds to the resting state of the pendulum. It is an attractor. In general, an attractor is a set of points in the phase space to which all trajectories are attracted [2].

Let us turn now to the driven pendulum. The driving of the pendulum isn't exactly sinusoidal because the light shaft is pulled up and down by the car. The measured angle vs. time diagram for a long interval (approx. 90 sec) of our driven pendulum is shown in Fig.4. Both the initial angle and angular velocity are zero $\varphi_0 = 0$, $\omega_0 = 0$. We can see indeed that the motion is not periodic.

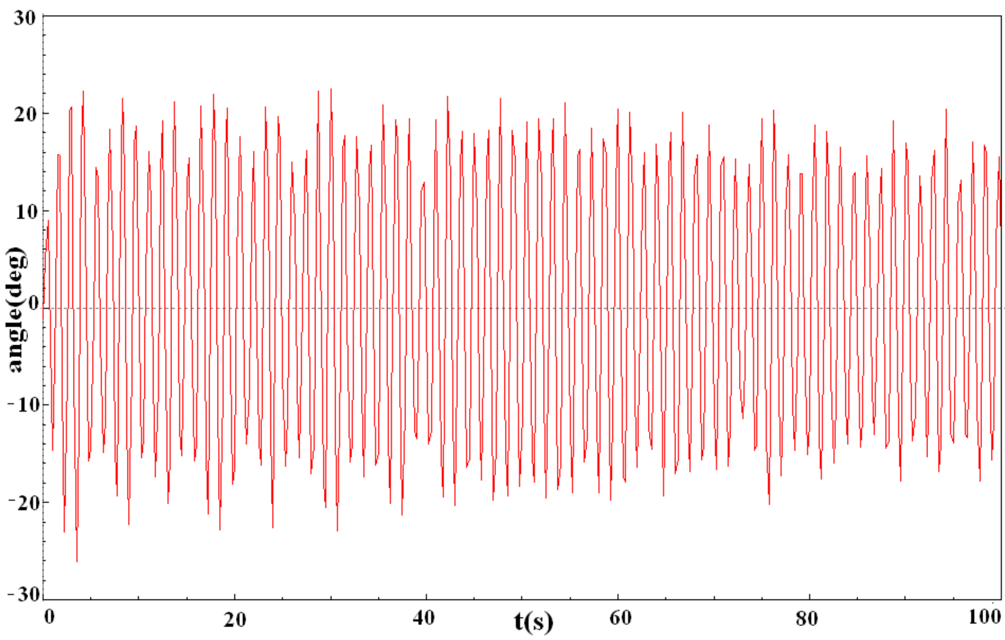


Figure.4: Measured angle vs. time graph for the driven damped pendulum, $\varphi_0 = 0; \omega_0 = 0$

Fig.5. exhibits the same motion in the phase space diagram with the same time interval and with the same initial conditions. The trajectory of the driven pendulum traces out a range between two ellipses. The trajectory is a coil, not a regular curve (as for the non-driven pendulum). This is a chaotic attractor, an attractor bound to non-periodic, chaotic motions [2].

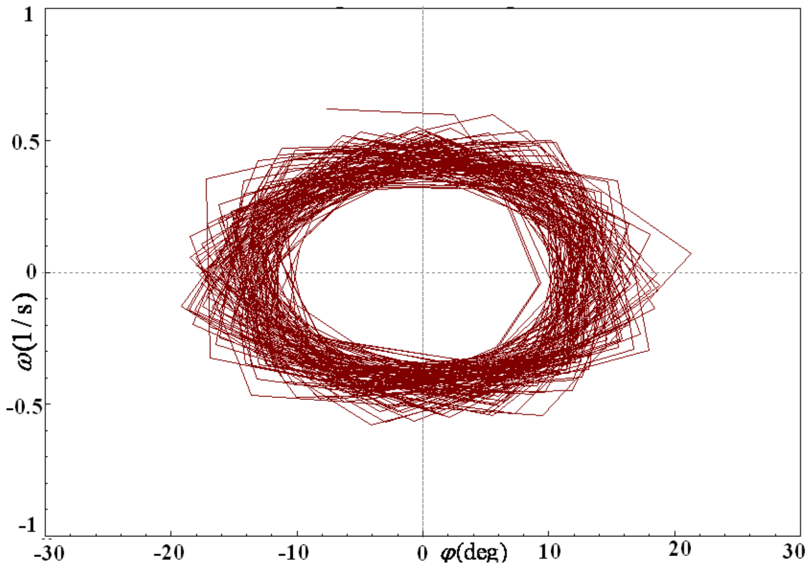


Figure.5: Phase space trajectory on the chaotic attractor of the driven pendulum of Fig.4.

An important characteristic of chaotic motions is the sensitive dependence on initial conditions. In order to show this experimentally, we started the pendulum twice at almost the same initial conditions. The initial angle and angular velocity were the same, zero, but the phase of the driving (the position of the shaft) was slightly different. We can see the phase space diagrams of the two motions: angular velocity versus angle trajectories in both cases (Fig.6.a and b). Trajectories run through a range between two ellipses. The initial form of the trajectories is markedly different, but the long-time behaviour is the same, corresponding to the fact that both motions belong to the same chaotic attractor.

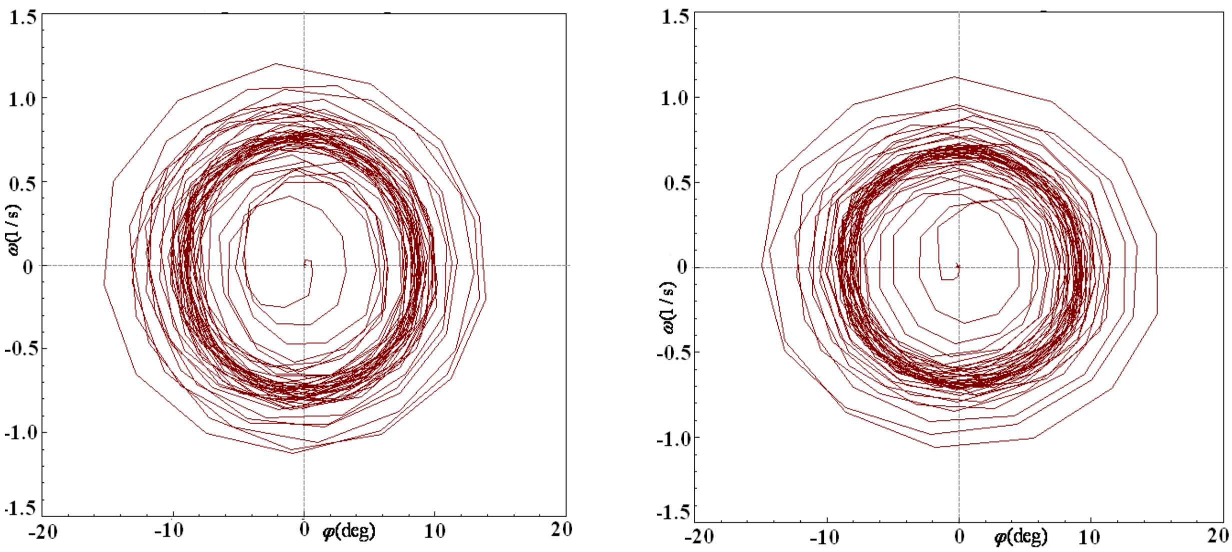


Figure.6. Phase space trajectories of the driven pendulum with the same initial positions $\varphi_0 = 0$; $\omega_0 = 0$, and slightly different phases. Data are extracted from the measured angle vs. time curves.

The angle difference between the two motions can be seen in Fig.7. The difference is growing for about 35 seconds, then the graph reaches a saturation.

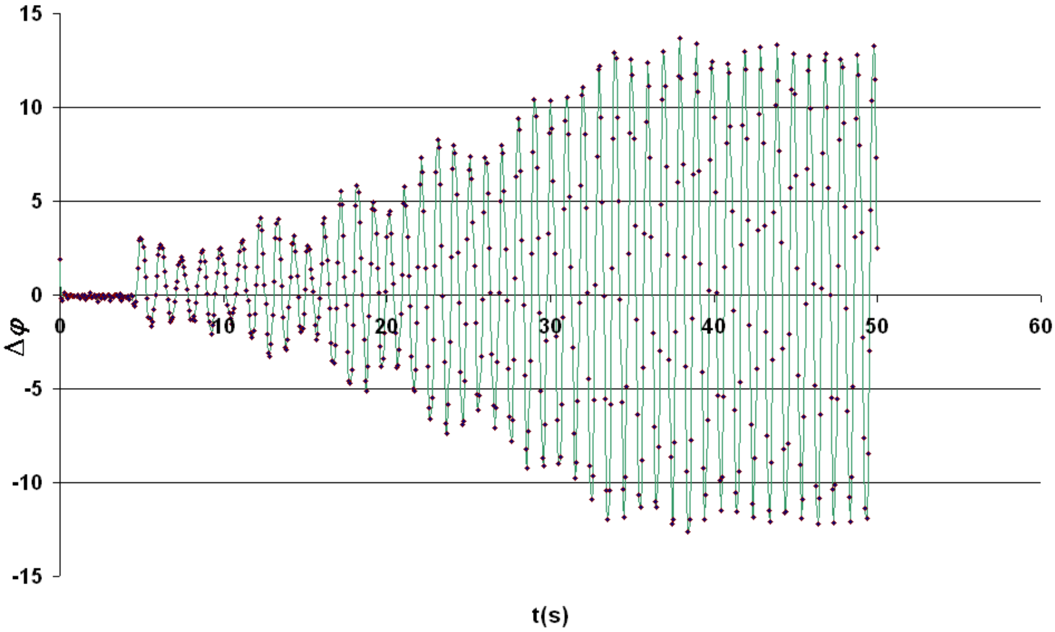


Figure.7: Angle difference vs. time for the measurement of Fig. 6.

The next diagram shows the absolute values of the angle differences in every half second. We tried to fit an exponential to the points and found a rule $e^{0.10t}$ (Fig.8.). Its positive exponent indicates the chaoticity of the motion. This exponent shows how fast the trajectories diverge. The logarithm of the fitted exponential is a straight line with a positive slope.

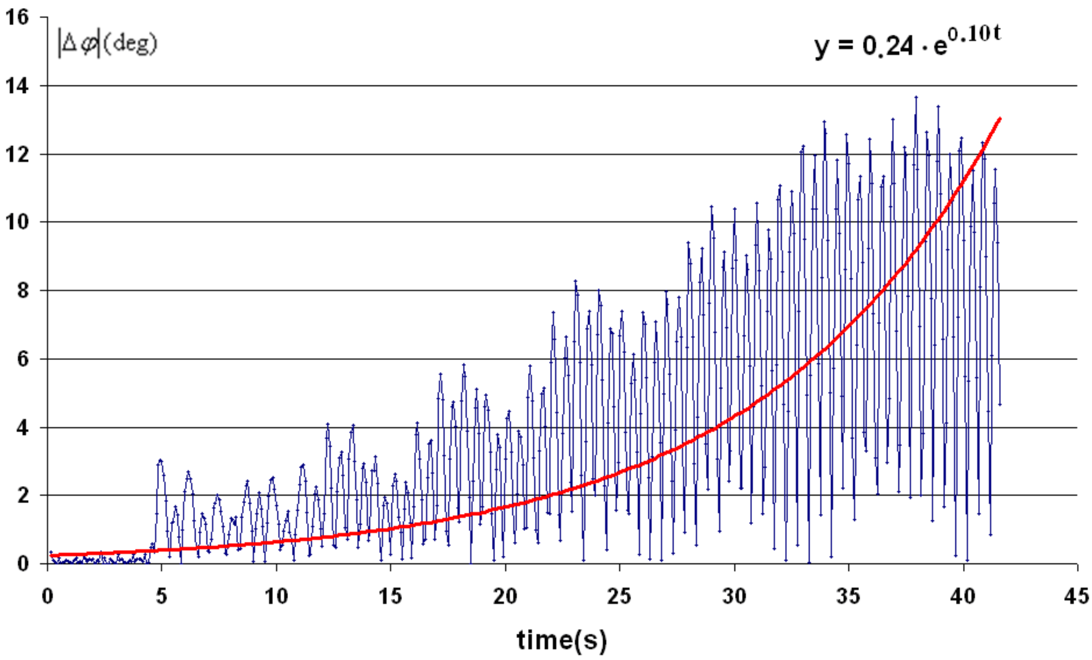


Figure.8: Exponential fit to the modulus of the angle difference $|\Delta\varphi|$ of Fig. 7 taken in every half second.

This slope is called the Ljapunov exponent, a quantitative measure of the sensitive dependence on the initial conditions. It is the averaged rate of divergence of two neighbouring trajectories. In general, the Ljapunov exponent is defined as [2]

$$\lambda = \frac{1}{t} \cdot \ln \frac{dr(t)}{dr(0)}$$

where *dr(t)* is the distance between two neighbouring trajectories in the phase space and *dr(0)* is the initial distance. The Lyapunov exponent can also be considered as the measure of the growth of uncertainty. Our measured exponent is $\lambda = 0.1$ (1/s).

In the knowledge of the Lyapunov exponent, the time of predictability can be estimated as [2,3]

$$t\_p \approx 1/\lambda.$$

In our measurement t_p = 10 sec. This implies that one can predict the motion only approx. 10 seconds in advance. Long term prediction is impossible!

3. Numerical simulations

We also tried to simulate the equation of motion numerically. The equation of motion for the angle φ of a pendulum whose suspension is moving along a horizontal line periodically with amplitude A and period T is from [3]:

$$\ddot{\varphi} = -\alpha \cdot \dot{\varphi} - \frac{g}{l} \cdot \sin\varphi + \frac{A}{l}\left(\frac{2\pi}{T}\right)^2 \cdot \cos\varphi \cdot \cos\left(2\pi \cdot \frac{t}{T}\right) \varphi$$

Here φ is the angle measured from the vertical axis (dot marks time derivative), *l* is the length of the pendulum, α denotes the damping constant, and *g* is the gravitational acceleration. Note that the non-driven problem corresponds to the choice A=0. This equation does not correspond exactly to our experiment since the driving is not entirely sinusoidal there, and the drag is presumable more complicated than the one proportional to the angular velocity.

Nevertheless, we run the program with the parameters fitted to our set-up as well as possible. The length of our pendulum is 0.38m, the driving amplitude is 0.1m, and its time period is 1.05s. Thus,

$$l = 0.38\text{m}, \ A = 0.1\text{m}, \ T = 1.05\text{s}.$$

Since the damping constant cannot be measured directly, we changed the damping constant between 0.001 (1/s) and 0.06 (1/s) in the simulations.

In the computer program, we applied a simply numerical scheme, called the leap-frog algorithm [4], to solve the equation of motion. The angle at time *t+dt* is obtained as

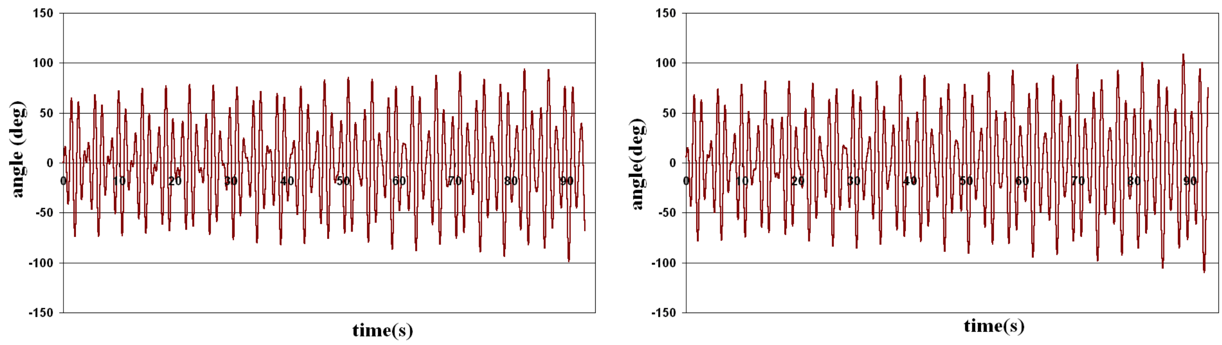
$$\varphi(t + dt) = \varphi(t) + dt \cdot \dot{\varphi}(t + dt/2)$$

while

$$\dot{\varphi}(t + dt/2) = \dot{\varphi}(t - dt/2) + dt \cdot \ddot{\varphi}(t)$$

for the angular velocity. We can get the initial angular velocity $\ddot{\varphi}$ (t=0) from the right hand side of the equation of motion with the initial values $\varphi_0 = \varphi(t=0)$, $\omega_0 = \dot{\varphi}(t=0)$. The accuracy of this algorithm is of order dt^3 in one time-step. We used a time-step dt = 0.0006 sec. (The program was written in Excel by an 18 y/o secondary grammar school student.)

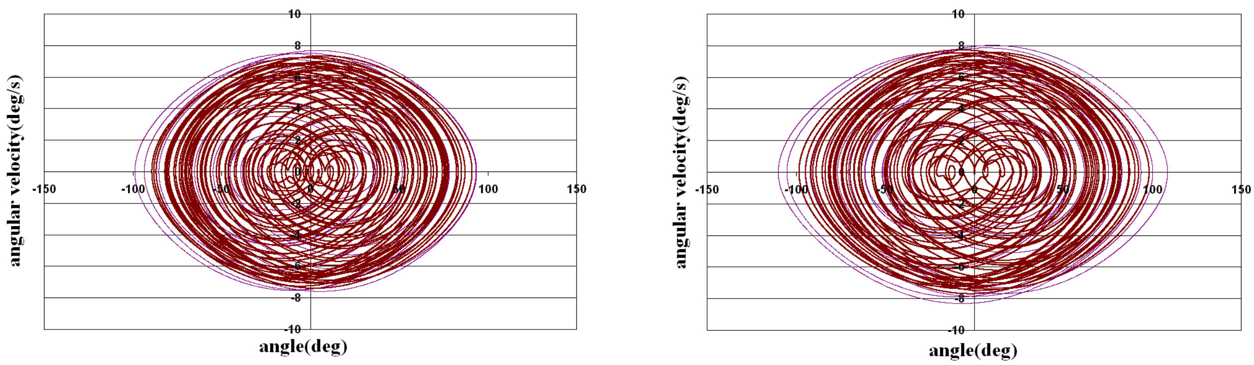
Fig.9. shows some of the results. The obtained angles vs. time graphs are shown with two slightly different initial conditions in Fig.9. (a) and (b). The graphs are qualitatively similar to the measured ones.



(a) (b)

Figure.9.: Simulated angle vs. time with T =1.05s ; $\omega_0 = 0\,\frac{1}{s}$; α = **0.001** $\frac{1}{s}$; $\varphi_0 = 0^0$ **(a)** $\phi_0 = 6^0$ **(b)**

We can see some difference between the trajectories in the phase space in Fig.10.a, b, particularly in the initial period, before reaching the chaotic attractors.



(a) (b)

Figure.10: The angular velocity vs. angle trajectory for the two cases of Figure.9.

The distance *dr* between two trajectories in the phase space can be defined as

$$dr = \sqrt{(\phi_1 - \phi_2)^2 + (\omega_1 - \omega_2)^2}$$

where φ_1 and φ_2 are the angles, and ω_1 , ω_2 the angular velocities for the two neighbouring trajectories. We determined the graph of the phase-space distance dr(t) for the two trajectories of Fig. 9 numerically, and fitted an exponential to it (Fig.11.)

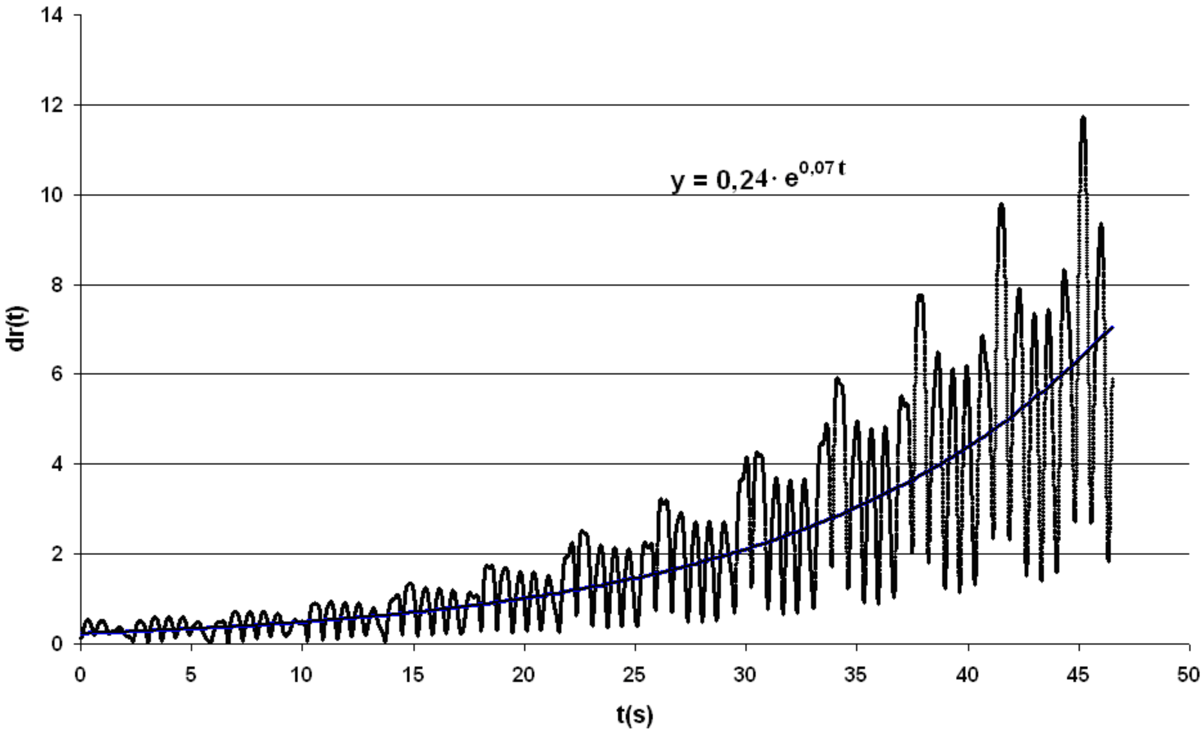


Figure.11.: Phase space distance with the data of Fig. 9 and an exponential fit. The smooth curve is of the form of $0.24 \cdot e^{0.07t}$

The Lyapunov exponent extracted from the fitted curve is $\lambda = 0.07$ (1/s) and so the time of predictability is t_p=13.7 sec at this damping ($\alpha = 0.001$ 1/s). t_p is on the same order as the measured value (11.1 sec), so we can conclude that the damping constant is about 0.001 1/s in our set-up.

The investigation of the full range (0.001, 0.06) 1/s of the damping constant leads to the conclusion that the time of predictability, t_p, grows monotonically from 11 to 54 sec. Qualitatively this means that increasing dissipation provides somewhat better predictability, but exponential divergence of nearby trajectories remains valid, and the system is unpredictable over long times even for as strong a damping as 0.06 1/s.

4. Conclusions

We have shown that high-school students can be motivated to experimentally investigate the motion of non-linear systems, like, e.g., that of a driven damped pendulum. They showed, via computer assisted measurements, that the dynamics is chaotic, i.e., unpredictable over long times. Even a characteristic measure of this, the Lyapunov exponent or the time of predictability, can be extracted from the data.

The most motivated students are able to learn the numerical simulation of Newtonian equations of motion. A known model of the experimentally investigated driven pendulum was simulated and led to a reasonable agreement between measurement and simulations. A full agreement cannot be expected since the driving of the measured pendulum is not exactly periodic, and, moreover, a linear damping can be a faithful model of the friction at the suspension axis of the pendulum only; the drag of the swinging body should be proportional to the square of its velocity measured relative to the standing air. A detailed investigation of the

sensitivity to the initial conditions would require taking an average over initial conditions [2,3]. We are not capable of investigating large statistics,
the order of magnitude agreement of the time of predictability for a typical initial condition is, therefore, satisfactory.

On a more general level, the most important effects of our investigations for the students are that they

- improve their experimental skills,
- investigate real problems and understand chaotic, unpredictable systems,
- learn to create phase space diagrams and understand their meaning,
- acquire skills in collecting and analysing data via computer,
- learn to solve differential equations numerically.

Some of them want to be engineers, physicists (or teachers), programmers, as many of the former members of our workshop.

Acknowledgements

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