Irregular Chaos in a Bowl

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Abstract
The investigation of chaotic systems itself is very interesting but the subject could be an excellent didactic tool in raising the interest of the students and motivate them to learn modern physics. The study of relatively simple chaotic systems can provide a deep insight into the deterministic and probabilistic behaviour of the natural processes already at introductory physics courses. The joint presentation of a real system and its mathematical model helps effectively the students to understand the intricate concepts and ideas used for the description of the physics of chaotic motion. In the present paper the dynamic behaviour of a ball moving in a complex-shaped bowl will be studied. The equations of motion can be solved by the freely downloadable Dynamics Solver program which is a well-accomplished tool for the investigation of dynamic systems[1].

Keywords
Chaotic motion, permanent and transient chaos, fractal basins.

Introduction

Computers have opened up a new dimension for physics experimentation. A completely new method, the computerized experimental physics (numerical simulations) has been developed for the quantitative investigation of those systems that previously could be studied only qualitatively. One of the most important and best-known fields of the numerical simulations is the study of chaotic systems. The investigation of chaotic systems itself is very interesting but the subject could be an excellent didactic tool in raising the interest of the students and motivate them to learn modern physics. The study of relatively simple chaotic systems can provide a deep insight into the deterministic and probabilistic behaviour of the natural processes already at introductory physics courses.

Although excellent introductory monographs are available which explain the basic ideas and concepts [Tél, T., Gruiz, M. (2006).], and in which a wide variety of simple mechanical systems producing chaotic behaviour are deployed [Korsch, H. J. and Jodl, H.-J. (1998).] [Gutzwiller, M. C. (1990).] [Szemplinska-Stupnicka, W. (2003).], it is worth searching for further simple mechanical systems which can be built easily and exhibit chaotic behaviour. The joint presentation of a real system and its mathematical model helps the students to understand effectively the intricate concepts and ideas used for the description of the physics of chaotic motion. In this paper besides the well-known forms of the chaotic motion the recently studied transient chaos which is exhibited in dissipative systems will be also demonstrated. It will be shown, that transient chaos is a good tool for the demonstration of the fractal geometry too.

The mechanical model

In the present paper the dynamic behaviour of a ball moving in a complex-shaped bowl will be studied. The shape of the bowl is defined by a height function *z(x,y)* of the points of the bowl [Tél, T., Gruiz, M. (2006)] [Márton, G., Tél, T. (2005)]. This function can be identified with the gravitational potential for the moving ball; therefore the equations of motion of the ball can be easily described. To approach the real motion the equations can be completed with a term of friction. If this term is zero, the motion is conservative; in other cases it is dissipative. It should be mentioned that if the *z(x,y)* function is given, then the bowl can be fabricated by a rapid prototyping procedure and the real motion of the ball can be also studied.

The motion of a point-like body of unit mass which is moving in a *V(x,y)* potential field under the influence of friction which is proportional with the velocity can be described by:

$$\ddot{x} = -\frac{\partial V}{\partial x} - \alpha\dot{x}, \quad \ddot{y} = -\frac{\partial V}{\partial y} - \alpha\dot{y}, \qquad (2.1)$$

[1] According to the authors Figures of high resolution are extremely important for the illustration of the results of the simulations. Unfortunately the size of the files submitted was limited, therefore the quality of Figures presented here is not good enough. A version of the paper with good quality Figures can be downloaded from: http://csodafizika.hu/ballinbowl/paper.pdf

where α is the coefficient of friction. If α=0 then the motion is conservative. Introducing the coordinates of the velocity as new variables, the equations can be transformed into the usual form:

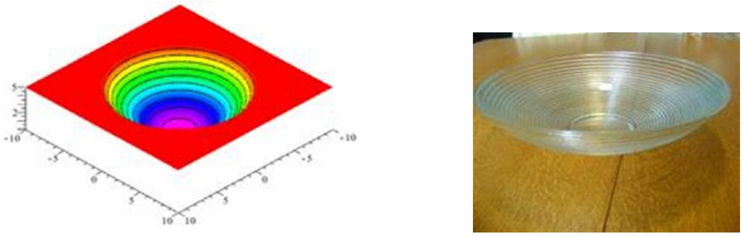
$$\left.\begin{aligned} \dot{x} &= f_1(x,y,u,v) = u \\ \dot{u} &= f_2(x,y,u,v) = -\frac{\partial V}{\partial x} - \alpha u \\ \dot{y} &= f_3(x,y,u,v) = v \\ \dot{v} &= f_4(x,y,u,v) = -\frac{\partial V}{\partial y} - \alpha v \end{aligned}\right\} \qquad (2.2)$$

The motion was studied in three different shaped bowls that is in three different gravitational potential fields. The height functions of the bowls are (x, y and z are measured in cm):

$$z(x,y) = V(x,y) = 0.1\cdot\left(x^2 + y^2 - 1\right) \qquad (2.3.\ a)$$

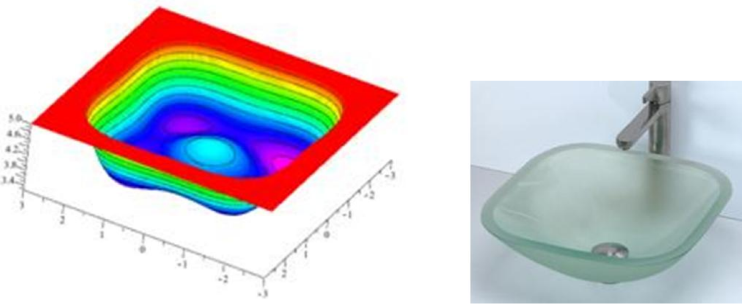
$$z(x,y) = V(x,y) = 0.1\left(x^4 + y^4 + 0.5x^2y^2 - 4x^2 - 4y^2 - 0.5xy + 40\right) \qquad (2.3.\ b)$$

$$z(x,y) = V(x,y) = 10^{-4}\left(x^4 + 9y^4 + 26x^2y^2 - 100x^2 - 300y^2 + 5000\right) \qquad (2.3.\ c)$$



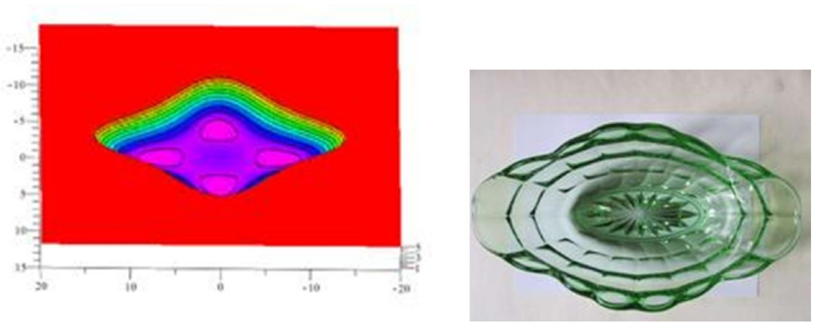
Maple display

Figure 1.(a) A bowl given by potential function (2.3.a) and a real bowl which is like the simulated one.



Maple display

Figure 1.(b) A bowl given by potential function (2.3.b) and a real bowl which is like the simulated one.



Maple display

Figure 1.(c) A bowl given by potential function (2.3.c) and a real bowl which is like the simulated one

It is important that while in case of the (2.3. a) potential the deepest point of the bowl is in its centre (x=0; y=0) in case of the other two potentials the deepest points of the bowls situate near the four peaks. The potential energy of the moving ball is the lowest at these points, therefore these are the stable equilibrium positions of the

ball. In case of potential function (2.3.b) these points are (x_1=1.3038; y_1=1.3038), (x_2=1.2247; y_2=-1.2247), (x_3=-1.3038; y_3=-1.3038) and (x_4=-1.2247; y_4=1.2247), while in case of the bowl with height function they are (x_1=0; y_1=4.0825), (x_2=0; y_2=-4.0825), (x_3=7.0711; y_3=0) (x_4=-7.0711; y_4=0).
The equations of motion can be easily solved with the *Dynamics Solver* [http://tp.lc.ehu.es/jma/ds/ds.html] program which is an ideal tool for the simulation of the dynamic systems. First the frictionless motion is presented. The total energy E for these motions is constant. The simplest case of the real motion if the ball is released from a point of the rim of the bowl without initial velocity (u_0=v_0=0). The results can be seen in Fig. 2.(a)-2.(c).

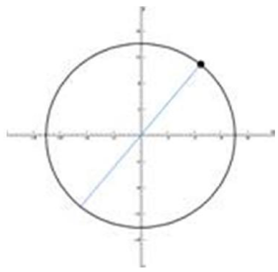


Figure 2(a) Motion in potential (2.3. a), E=20 (x_0=9; y_0=10.95445).

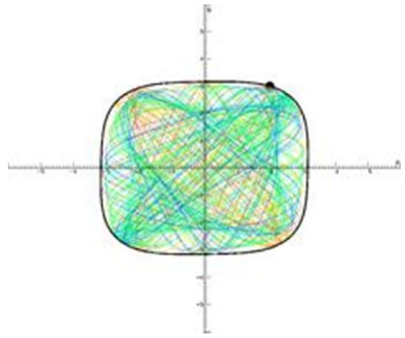


Figure2.(b) Motion in potential (2.3.b), E=10 (x_0=2; y_0=3).

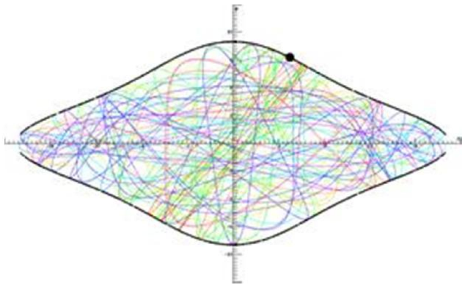
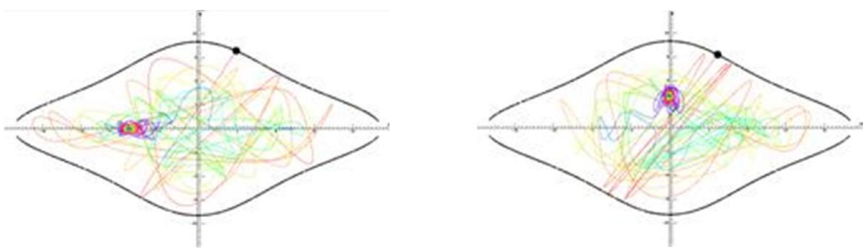


Figure 2.(c) Motion in potential (2.3.c), E=10 (x_0=5; y_0=9.2796).

In case of potential (2.3.a) a simple periodic motion is occurring, while in case of potentials (2.3.b), and (2.3.c) chaotic motion appears. Fig. 2.(b) and 2.(c) show that the envelope of the trajectories of the frictionless motion give back the shape of the bowls. In Fig. 3 the results of simulations of dissipative motion (α=0.005) in potential (2.3.c) with two different initial conditions can be seen.



(x_0=4; y_0=9.86) (x_0=5; y_0=9.2796)

Figure 3. Trajectories of frictional motion in potential (2.3.c).

It can be seen that trajectories (as it was expected) after some chaotic part tend to one minimum points of the potential well. Such type of behaviour is called *transient chaos* because the chaotic motion is interim and after it the motion becomes periodic or it stops. In our case the attractors where the motion ends up are the minimum points of the potential well.

One of the most important feature of the chaotic motion is its extremal sensitivity to the initial conditions. This can be demonstrated very attractively with the so called spaghetti-plots. The diagram shows a characteristic of a system as a function of time when its motion is starting from nearby initial conditions. (In case of the ball moving in a bowl this characteristic can be a coordinate of the ball.) A typical spaghetti-plot is really similar to a torch light. The trajectories of the motion with slightly different initial conditions deviate soon strongly and the diagram shows that the motion of the ball totally unpredictable.

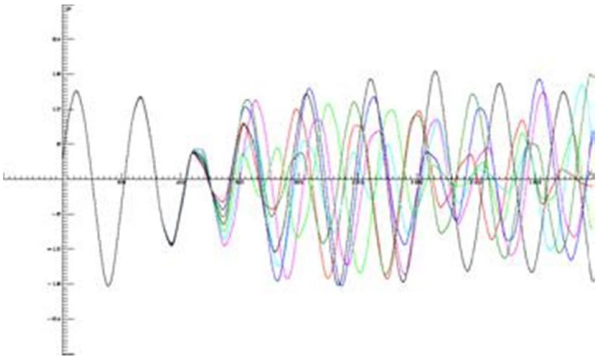


Figure 4. Spaghetti-plot with seven slightly different initial x_0 (E=10, y_0=5, v_0=0).

In Fig. 4 the x coordinate of a ball which is moving in a bowl of potential (2.3.c) is plotted as a function of time at seven nearby initial conditions (x_0=2.97, 2.98, 2.99, 3.00, 3.01, 3.02 and 3.03) The trajectories run together till the time t=50, but after it they strongly deviate. So this motion can be predicted only till this time. For longer periods only the probability that the ball comes to a small neighbourhood of a given point can be determined. However, the motion of the ball is conservative, so the possible values of the x coordinates are restricted by the parameter E (the total energy of the ball), therefore the torch does not open fully

Frictionless motion

The total energy of the moving ball is at any instant the sum of the kinetic and potential energy:

$$E(x,y,u,v)=\frac{1}{2}\left(u^2+v^2\right)+V(x,y). \qquad (3.1)$$

To study chaotic dynamics the Poincare map is a very useful tool. A Poincare map can be interpreted as a discrete dynamical system with a state space that is one dimension smaller than the original continuous dynamical system.

Although the phase space of the moving ball is a four dimension one *(x,y,u,v)* one can study the system and can get a good picture about its behaviour by the use of a two dimensional Poincare map. In the frictionless case there are no attractors the character of motion depends on the initial conditions. In order to get an overview of the system's behaviour Poincare maps belonging to the same energy, but corresponding to different initial conditions should be plotted. From the four initial conditions (x_0,y_0,u_0,v_0) only three can be chosen freely, the fourth one are determined by the energy equation (3.1).

Figure 5 shows the (x,y) Poincare map of the motion occurred in the bowl of potential (3.b) (simulations were made by dynamic solver.)

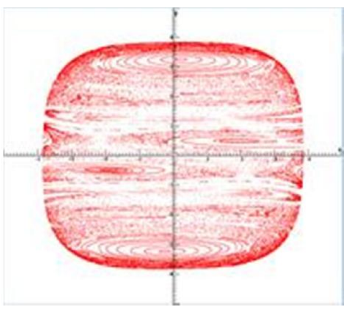


Figure 5. (x, y) Poincare map for motion in potential (3.b) (E=20).

It is a typical map for conservative chaos there are big fat fractal like areas with periodic isles in them. Similar plots can be obtained in the (x-u) and (y-v) Poincare maps.

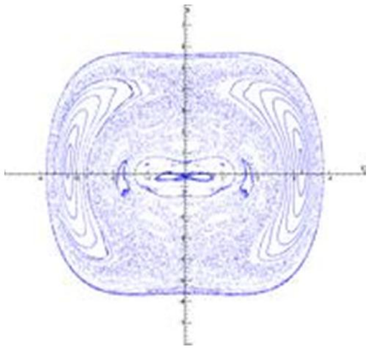


Figure 6. (x-u) Poincaré map for motion in potential (3.b) (E=20).

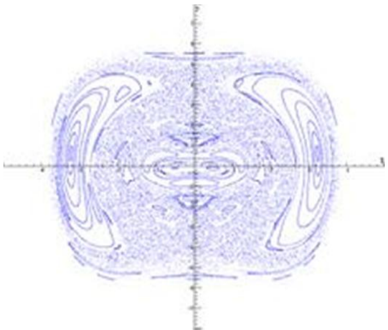
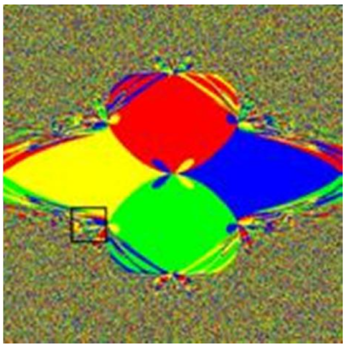


Figure 7. (y-v) Poincaré map for motion in potential (3.b) (E=20)

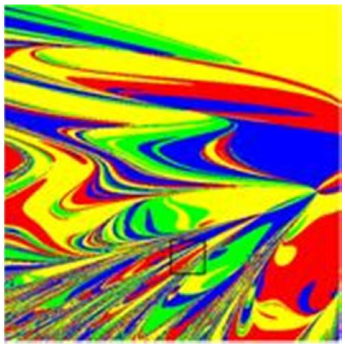
Frictional motion

As it was mentioned earlier the ball comes to rest in a well near one peak of the bowl. In case of frictional motion the first period of the motion can exhibit transient chaos.
In the following the structure of the basins of attraction for the motion occurring in bowl (in potentials) which can be seen in Figure 1(c) will be revealed. A basin of attraction is the set of the initial positions whence the orbits of the balls released with zero initial velocity tend to the same attractor. The potential characterising the bowl has four potential well which were determined earlier. In the maps of the figures shown below the potential wells were marked with different colours and their basins of attractions was painted by the same colour too. The colours belonging to the attractors (x_1=0; y_1=4.0825), (x_2=0; y_2=-4.0825), (x_3=7.0711; y_3=0), and (x_4=-7.0711; y_4=0) are red, green, blue and yellow, respectively. Figures show the structure of the basins developed if the friction coefficient was α=0.01. The sequence of pictures arranged alphabetically in the figures. Every picture consists of 500×500 points and in every one a small square is chosen at a boundary of either basins. Every member of the sequence of the pictures shows the ten times magnified image of the square marked in the preceding picture.
As it can be expected the boundaries of the basins exhibit fractal structure. The boundaries of these *basins show fractal geometry* which can be described by a very complicated structure like a Cantor set. In other words, whenever two basins seem to meet, we discover upon closer examination that a third basin is there in between them, and so ad infinitum.

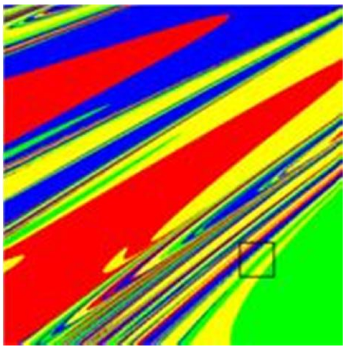
In table1the fractal dimension of boundaries between the attraction basins shown in Figure 8.(c) and Figure 9.(c) are given as a function of the magnification.



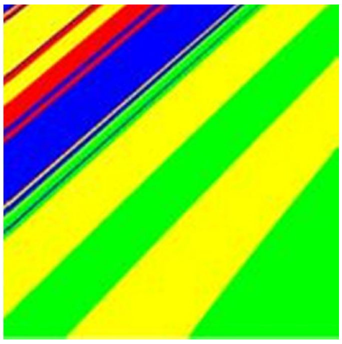
(a) $x, y \in [-10;10]$



(b) $x \in [-6;-4], y \in [-4;-2]$

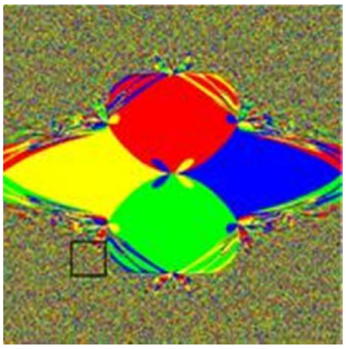


(c) $x \in [-5;-4.8], y \in [-3.6;-3.4]$

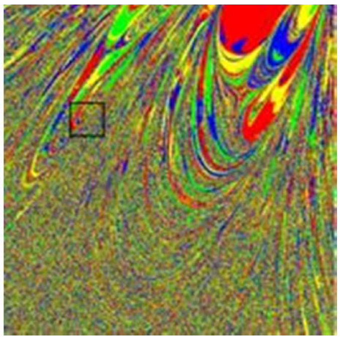


(d) $x \in [-4.86;-4.84], y \in [-3.56;-3.54]$

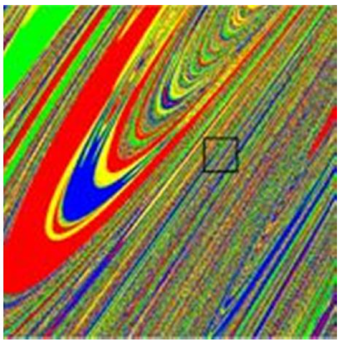
Figure 8. Basins of attraction for the bowl of potential (2.3.c) (Friction coefficient: $\alpha = 0,01$, resolution: 500×500 initial velocity: 0).



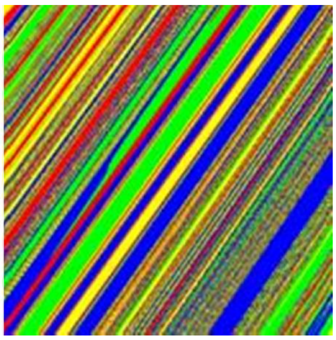
(a) $x, y \in [-10;10]$



(b) $x \in [-6;-4], y \in [-6;-4]$



(c) $x \in [-5.6;-5.4], y \in [-4.8;-4.6]$



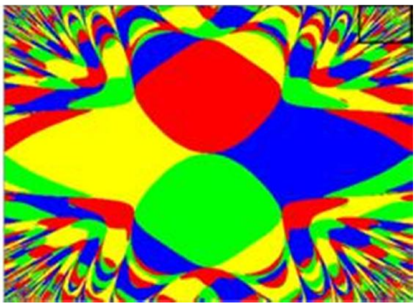
(d) $x \in [-5.48;-5.46], y \in [-4.7;-4.68]$

Figure 9. Basins of attraction for the bowl of potential (2.3.c) (Friction coefficient: $\alpha = 0,01$, resolution: 500×500 initial velocity: 0).

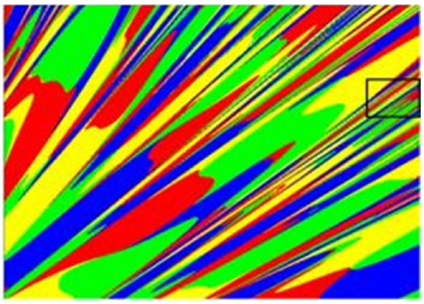
Table 1. Fractal dimension of boundaries between the attraction basins are given as a function of the magnification

	$x \in [-5;-4.8], y \in [-3.6;-3.4]$	$x \in [-5.6;-5.4], y \in [-4.8;-4.6]$
50×50	1.30	1.55
100×100	1.26	1.51
200×200	1.23	1.44
400×400	1.19	1.33
800×800	1.16	1.29

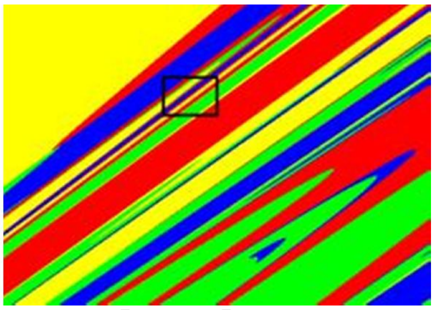
The study of transient chaos can be a very important didactic tool in the demonstration of fractal geometry [Ying-Cheng, L., Tél, T. (2011).]. Generally fractals appear in the phase space of the chaotic systems. Fractals in an abstract space are sometimes not expressive and meaningful to students. In contrast to this the attraction basins of the attractors of transient chaos exhibit in the real space, so students can understand the properties of the fractals a more suggestively in the real geometric space. The development of the fractal boundaries are illustrated well by the video [http://indavideo.hu/video/Magneses_inga_fraktal_vonzasi_tartomanyai] showing the real and simulated motion of a magnetic pendulum.
The fractal basin boundaries have shown *irregular behaviour: in this case the classical parameters used to describe chaos became time dependent and the structure of the basins was not fully invariant upon magnification*. The measured dimension of the basin boundaries can be non-integer over all finite scales, but have asymptotic fractal co-dimension: one. This phenomenon is recently referred as *doubly transient chaos* [Motter, A. E., Gruiz, M., Karolyi, Gy., Tel, T. (2013).].
In case of higher friction the picture of the attraction basins is slightly different from the previous one (Figure 10).



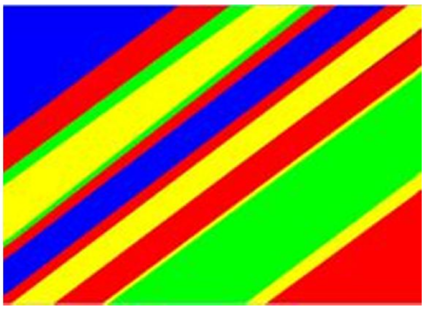
(a) $x \in [-11;11], y \in [-8;8]$



(b) $x \in [8,25;11], y \in [6;8]$



(c) $x \in \left[10\frac{42}{64};11\right], y \in [7.25;7.5]$



(d) $x \in \left[10\frac{402}{512};11\right], y \in \left[7\frac{13}{32};7\frac{14}{32}\right]$

Figure 10. Basins of attraction for the bowl of potential (2.3.c) (Friction coefficient: 0.05, resolution: 550×400 initial velocity: 0).

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