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Gravity Probe B experiment and gravitomagnetism

B Vető

Department of Physics of Roland Eötvös University, Pázmány Péter sétány 1/A. Budapest H-1117, Hungary

E-mail: veto.balazs@metal.elte.hu

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Abstract

Gravitomagnetism is a low velocity and weak gravitational field approximation of general relativity. It provides a simple approach to post-Newtonian gravitational phenomena via electromagnetic analogy. Intended for advanced undergraduate students, the present paper applies gravitomagnetism to the quantitative study of the geodetic precession and frame dragging effects, both verified recently in the Gravity Probe B experiment. The result obtained is identical with Schiff's formula.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Gravity Probe B (GP-B) experiment and its results have been published recently and are now widely known due to the internet, see Everitt *et al* (2008). Because of its popularity and scientific significance, the GP-B experiment should be a part of the curriculum on relativity for advanced undergraduate students with a special interest in this field. The geodetic precession and frame dragging effects investigated in the GP-B experiment are, however, difficult to teach at undergraduate level, because their derivation requires the knowledge and mathematical toolbox of general relativity. In this paper, a new methodology is presented to help students understand the physical origin of both effects. This alternative methodology is based on gravitomagnetism, which is an approximation of general relativity at low velocities and weak gravitational fields. The gravitomagnetic approximation describes both effects correctly without using the more complicated formalism of general relativity. Pointing out the close analogy between electromagnetism and gravitomagnetism, the methodology builds on the students' previously acquired knowledge of electromagnetism. The usefulness of this alternative methodology is that even students unfamiliar with general relativity can gain insight into the physics of geodetic precession and frame dragging effects.

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The underlying principle of the gravitomagnetic interpretation of geodetic and frame dragging effects is that the gravitomagnetic field exerts a torque on a gravitomagnetic dipole, i.e. on rotating bodies. This is analogous to the effect of an electromagnetic field exerting a torque on a magnetic dipole (e.g. circular conductor or compass). The gyroscopes in the GP-B experiment are gravitomagnetic dipoles that experience a torque caused by the Earth's gravitomagnetic field. The spin axes of the gyroscopes change direction due to the applied torque.

2. The Gravity Probe B experiment

The geodetic and frame dragging effects were stipulated shortly after the publication of Einstein's theory of general relativity. The experimental measurement of geodetic and frame dragging effects took place only decades later with the GP-B experiment. The GP-B experiment's equipment consisted of a set of four gyroscopes housed in a satellite set on a circular polar orbit. The direction of the spin axes of gyroscopes was aligned with the orbit plane. The experiment's goal was to measure the change in orientation of the spin axis of the gyroscopes with unprecedented precision in two perpendicular directions: (1) the geodetic effect due to orbital motion—in the plane of the orbit, (2) the frame dragging effect due to the Earth's rotation—perpendicular to the orbit plane.

The angular velocity of the precession of the spin axis of a gyroscope orbiting the Earth is described relative to the position of fixed stars by Schiff's formula, Schiff (1960). In the case of the GP-B experiment, Schiff's formula predicts an angular velocity for the geodetic precession

$$\Omega_{\mathrm{G}} = \frac{3G}{2c^2}\frac{M}{R}\omega_{\mathrm{K}} = 6.606 \text{ arcsec/year}, \tag{2.1}$$

and averaged around the orbit, the angular velocity of precession for the frame dragging effect

$$\Omega_{\mathrm{FR}} = \frac{G}{2c^2}\frac{I}{R^3}\omega_{\mathrm{E}} = 0.039 \text{ arcsec/year, respectively.} \tag{2.2}$$

Here, Ω_{G} and Ω_{FR} are the angular velocities of geodetic and frame dragging precession, G is the gravitational constant, M and I are the mass and moment of inertia of the Earth, R is the orbital radius, ω_{K} is the orbital angular velocity of the satellite and ω_{E} is the angular velocity of the Earth's rotation.

3. The Thomas precession

Let us imagine an ideal gyroscope orbiting at a uniform angular velocity ω_{K} along a circle of radius r. The gyroscope is ideal and there is no net torque acting on it. The spin axis of an ideal gyroscope at rest in an inertial frame points steadily to a given direction. In our example, the gyroscope is moving along an orbit and it is at rest in a rotating, i.e. accelerated, frame. The spin axis of this accelerated gyroscope precesses backwards in the plane of the circular orbit when observed from an inertial frame. This relativistic phenomenon is called the Thomas precession and it was first published in Thomas (1926).

Below we present a simplified explanation for the Thomas precession, following Mashhoon *et al* (2001). For the observer in the rest frame of the gyroscope, the spin axis of the gyroscope precesses backwards with an angular velocity ω_{K}. Without time dilation, the spin direction remains fixed in the inertial frame. In reality, however, the proper time τ in the rotating frame is $\mathrm{d}\tau = \mathrm{d}t(1 - v^2/c^2)^{1/2}$, where $v = \omega_{\mathrm{K}} r$. Hence, the backward rotation of spin axis occurs with respect to the rotating observer's proper time with an angular velocity

$\omega = \omega_K(dt/d\tau)$. From the standpoint of the inertial observer, the spin direction precesses with $\Omega_T = \omega_K(1 - dt/d\tau)$ in the orbit plane.

If the gyroscope is set on the board of a satellite orbiting the Earth and the satellite's Keplerian orbital angular velocity vector is ω_K, the gyroscope starts to precess with

$$\Omega_T = \omega_K[1 - (1 - v^2/c^2)^{-1/2}] \tag{3.1}$$

compared to the fixed stars, where v is the satellite's velocity along the orbit. If $v \ll c$, then the Thomas precession to the first order to v^2/c^2 is

$$\Omega_T = -\frac{v^2}{2c^2}\omega_K. \tag{3.2}$$

Considering that a body orbiting around a centre of gravity is subject to the virial theorem, i.e. its kinetic energy is half of its gravitational potential energy, $v^2 = GM/R$, then the Thomas precession of the gyroscope is

$$\Omega_T = -\frac{GM}{2Rc^2}\omega_K. \tag{3.3}$$

The Thomas precession is independent of gravity and it takes place when a gyroscope is in a circular (or other accelerating) motion. In the case of a gyroscope orbiting the Earth, both the Thomas precession and the gravitomagnetic effect of the Earth have to be considered while describing the change of direction of the gyroscope's spin axis.

4. Gravitomagnetism

Gravitomagnetism is a linear, relativistic approximation of gravity at low velocities and in weak gravitational fields ($v^2, \Phi \ll c^2$, where Φ is the gravitational potential) in general relativity. The phenomenon of gravitomagnetism (or gravitoelectromagnetism, as often called) is discussed in details in textbooks, see for example Wald (1984). The fundamentals of the gravitational and gravitomagnetic fields are given by the gravitational Maxwell equations, as derived rigorously in the paper of Hughes (1994). The gravitomagnetic Maxwell equations (4.1)–(4.4) are taken from Mashhoon (2000), where $\boldsymbol{g}$ and $\boldsymbol{b}$ denote the gravitational and the gravitomagnetic fields, respectively. The electromagnetic Maxwell equations are presented in parentheses:

$$\operatorname{rot}\boldsymbol{g} = -\frac{1}{2}\frac{\partial \boldsymbol{b}}{\partial t} \qquad \left(\operatorname{rot}\boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t}\right) \tag{4.1}$$

$$\operatorname{rot}\frac{\boldsymbol{b}}{2} = -\frac{4\pi G}{c^2}\boldsymbol{j}_m + \frac{1}{c^2}\frac{\partial \boldsymbol{g}}{\partial t} \qquad \left(\operatorname{rot}\boldsymbol{B} = \frac{4\pi k}{c^2}\boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{4.2}$$

$$\operatorname{div}\boldsymbol{g} = -4\pi G\rho_m \qquad (\operatorname{div}\boldsymbol{E} = 4\pi k\rho) \tag{4.3}$$

$$\operatorname{div}\frac{\boldsymbol{b}}{2} = 0 \qquad (\operatorname{div}\boldsymbol{B} = 0), \tag{4.4}$$

where $\boldsymbol{j}_m$ denotes the mass current density and ρ_m is the mass density in the given frame of reference. In equations (4.1)–(4.4) the analogy of electromagnetism and gravitomagnetism is significant, but there are remarkable differences, also. The gravitomagnetic Maxwell equations imply that the gravitomagnetic field is induced by moving masses, while the electromagnetic field is induced by moving charged particles. The signs of field strengths $\boldsymbol{g}$ and $\boldsymbol{E}$ are different because gravitational force is an attractive force, while electric force between two uniform charges is a repulsive one. Furthermore, $\boldsymbol{b}/2$ stands in the gravitomagnetic equations as the

analogous of $\boldsymbol{B}$, which indicates that the gravitomagnetic field $\boldsymbol{b}$ is induced by the twofold of mass currents and its effect on moving masses is duplicated also. As referred by Robertson (2005), the gravitomagnetic interaction is not Lorentz invariant.

A point mass, which is moving in a gravitomagnetic field, experiences a gravitomagnetic Lorentz force. This force is similar to the electromagnetic Lorentz force, acting on a charge which moves in an electromagnetic field. However, the gravitomagnetic and electromagnetic Lorentz forces differ by a factor of 2, in the form

$$\boldsymbol{F}_{\mathrm{GM}} = 2m\boldsymbol{v} \times \boldsymbol{b}, \qquad (\boldsymbol{F}_{\mathrm{L}} = q\boldsymbol{v} \times \boldsymbol{B}). \tag{4.5}$$

The gravitomagnetic field exerts a torque on a gravitomagnetic dipole moment, likely the electromagnetic field does it on a magnetic dipole moment,

$$\boldsymbol{M}_{\mathrm{GM}} = 2\boldsymbol{m}_{\mathrm{GM}} \times \boldsymbol{b}, \qquad (\boldsymbol{M}_{\mathrm{EM}} = \boldsymbol{m}_{\mathrm{EM}} \times \boldsymbol{B}), \tag{4.6}$$

keeping the difference of a factor of 2 between the gravitomagnetic and electromagnetic torques. The factor of 2 appears in equations (4.2) and (4.6), which results formally in a difference of a factor of 4 between the gravitomagnetic and electromagnetic torques exerted by mass and electric currents on the gravitomagnetic and electromagnetic dipoles, respectively.

The gravitomagnetic dipole moment vector is analogous to the electromagnetic dipole moment, defined by

$$\boldsymbol{m}_{\mathrm{GM}} = I_m \boldsymbol{A}, \tag{4.7}$$

where $\boldsymbol{A}$ is the vector of the area enclosed by the mass current I_{m}. By this definition, every rotating body represents a mass current and due to which it is a gravitomagnetic dipole. To determine the gravitational dipole moment of a gyroscope, let us consider a rigid body rotating around a rotational axis with an angular velocity of ω. Replacing $I_m = \omega\,\mathrm{d}m/(2\pi)$ and $A = r^2\pi$ in equation (4.7), the vector of the gravitomagnetic dipole moment of a mass element $\mathrm{d}m$ of this rotating body at a distance of r from the rotational axis is

$$\mathrm{d}\boldsymbol{m}_{\mathrm{GM}} = \frac{\omega}{2} r^2\,\mathrm{d}m. \tag{4.8}$$

The angular momentum of the same mass element is $\mathrm{d}\boldsymbol{S} = \omega r^2\,\mathrm{d}m$, i.e. the gravitomagnetic dipole moment and angular momentum are related by $\boldsymbol{S} = 2\boldsymbol{m}_{\mathrm{GM}}$, for any rotating body, independent of its shape and mass distribution. Using this relation in equation (4.6), the torque exerted by the gravitomagnetic field on a rotating body expressed in terms of the angular momentum is

$$\boldsymbol{M}_{\mathrm{GM}} = \boldsymbol{S} \times \boldsymbol{b}. \tag{4.9}$$

This is the effect of the gravitomagnetic field on gyroscopes.

5. Earth's gravitomagnetic field experienced by a satellite

From the reference frame of the satellite, the Earth is orbiting the satellite and rotating around its axis. For an observer on the board of the satellite, both motions of the Earth are mass currents, which induce gravitomagnetic fields in the satellite. Let us determine these fields. Using equation (4.2) and the fact that $\boldsymbol{g} = 0$ in the free falling satellite, we have

$$\mathrm{rot}\,\frac{\boldsymbol{b}}{2} = -\frac{4\pi G}{c^2}\boldsymbol{j}_{\mathrm{m}}, \tag{5.1}$$

From which we get the gravitomagnetic Biot–Savart law,

$$\mathrm{d}\boldsymbol{b} = -\frac{2GI_{\mathrm{m}}\,\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}}{c^2 r^3}, \tag{5.2}$$

needed to determine the Earth's gravitomagnetic field.

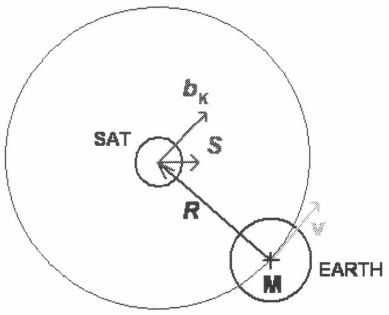

Figure 1. The Earth's mass current around the satellite. The direction of the gravitomagnetic field $\boldsymbol{b}_K$ induced by this mass current is perpendicular to the plane of the orbit and to the gyroscope's angular momentum vector $\boldsymbol{S}$, and it is pointing inwards the plane of the figure.

5.1. The gravitomagnetic field induced by the Earth's orbiting

Let us use the gravitomagnetic Biot–Savart law to determine the induced gravitomagnetic field experienced in the satellite due to the virtual orbiting of the Earth. The product of the elemental displacement $\mathrm{d}\boldsymbol{s}$ and the mass current I_m produced by the moving Earth can be replaced by $M\boldsymbol{v}$, by the product of the Earth's mass and its orbital velocity in equation (5.2). Let R to designate the radius of the satellite's orbit. (See figure 1.) The gravitomagnetic field inside the satellite induced by the mass current of the Earth is

$$\boldsymbol{b}_K = -\frac{2GM\boldsymbol{v} \times \boldsymbol{R}}{c^2 R^3}. \tag{5.3}$$

During the orbiting, $\boldsymbol{v} \times \boldsymbol{R}$ is constant, so can be expressed by $\boldsymbol{v} \times \boldsymbol{R} = \boldsymbol{\omega}_K R^2$, using the angular velocity vector. The Earth's orbiting is creating a gravitomagnetic field at the satellite which is constant in time:

$$\boldsymbol{b}_K = -\frac{2GM}{c^2 R}\boldsymbol{\omega}_K. \tag{5.4}$$

5.2. The gravitomagnetic field caused by the Earth's rotation

The rotating Earth is a gravitomagnetic dipole. The satellite orbiting the Earth is moving in the dipole field $\boldsymbol{b}_R$. Inside the satellite the gravitomagnetic field is continuously varying as the satellite is moving along its orbit. The Earth's gravitomagnetic dipole moment is

$$\boldsymbol{m}_{EA} = \frac{I\boldsymbol{\omega}_E}{2}. \tag{5.5}$$

Since the gravitomagnetic field $\boldsymbol{b}_R$ is induced by twice the dipole moment, the radial and tangential components of the field observed in the satellite in a position of the orbit around the Earth with radius R are

$$b_r = -\frac{2G}{c^2}\frac{I\omega_E}{R^3}\cos\vartheta, \tag{5.6}$$

$$b_\vartheta = -\frac{G}{c^2}\frac{I\omega_E}{R^3}\sin\vartheta, \tag{5.7}$$

where ϑ is the polar angle of the Earth's position in the polar coordinate system in the plane of the orbit, fixed to the satellite. (See figure 2.)

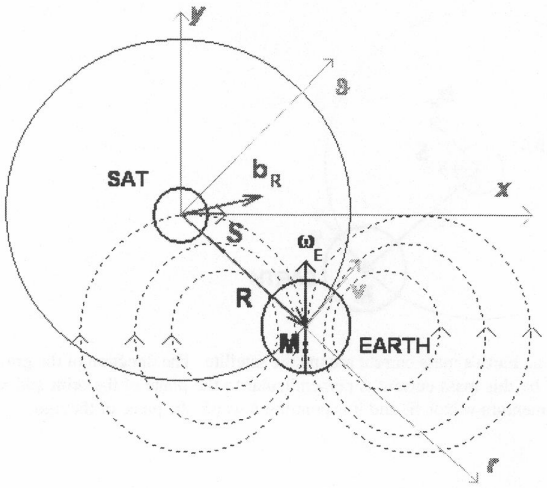

Figure 2. The instantaneous gravitomagnetic field b_R experienced by the satellite while crossing the gravitomagnetic dipole field of the rotating Earth. It is oriented in the plane of the orbit. The angular momentum of the gyroscope S is set to be perpendicular to the Earth's rotational angular velocity vector ω_E, like in the GP-B experiment.

The GP-B experiment measures the precession on a long time scale compared to one revolution of the satellite, so that the gravitomagnetic field strength can be averaged along the orbit. For averaging the gravitomagnetic field, let us transform the components of b_R into an orthogonal coordinate system fixed to the satellite as shown in figure 2:

$$b_x = \frac{3GI\omega_E}{c^2R^3}\sin\vartheta\cos\vartheta, \tag{5.8}$$

$$b_y = -\frac{GI\omega_E}{c^2R^3}(2\cos^2\vartheta - \sin^2\vartheta). \tag{5.9}$$

For the average field strength over the orbit $\overline{b}_x = 0, \overline{b}_y = -GI\omega_E/(2c^2R^3)$ is obtained. Since the y axis is parallel to ω_E, the average gravitomagnetic field in the satellite caused by the Earth's rotation is given by

$$\overline{b}_R = -\frac{GI}{2c^2R^3}\omega_E. \tag{5.10}$$

6. The gravitomagnetic and geodetic precession

Based on the laws of mechanics, a torque M acting on an ideal gyroscope will change the orientation of the gyroscope's spin axis. The angular velocity of the precession is

$$\Omega = \frac{S \times M}{S^2}. \tag{6.1}$$

If the torque is exerted by the gravitomagnetic field, then using equation (4.9) the gravitomagnetic angular velocity of the gyroscope's precession is

$$\Omega = \frac{S \times (S \times b)}{S^2}. \tag{6.2}$$

The angular momentum vector $\boldsymbol{S}$ of the gyroscope inside the satellite is perpendicular to the gravitomagnetic field b_K induced by the satellite's orbital motion. Using equation (6.2) due to the orthogonality of the two vectors we have

$$\Omega_{GM} = -b_K, \tag{6.3}$$

meaning that the angular velocity vector of the precession caused by the orbital motion has the same magnitude but opposite direction to the gravitomagnetic field. Using equation (5.4) for the gravitomagnetic field, we get the angular velocity of the gravitomagnetic precession of the gyroscope's spin axis as

$$\Omega_{GM} = \frac{2GM}{c^2R}\omega_K. \tag{6.4}$$

The spin axis of the gyroscope precesses in the orbit plane, into the same direction as of the orbital motion of the satellite. Since the satellite is orbiting the Earth, the gyroscope is also undergoing the Thomas precession according to equation (3.3). The Thomas precession also takes place in the orbit plane but opposite the orbital velocity. In the reference frame of the fixed stars, the precession of the spin axis of the gyroscope is the geodetic precession, which is the sum of the gravitomagnetic and Thomas precession, $\Omega_G = \Omega_{GM} + \Omega_T$, or

$$\Omega_G = \frac{3GM}{2c^2R}\omega_K, \tag{6.5}$$

in agreement with Schiff's formula in equation (2.1). This total angular velocity of the precession is anticipated by general relativity, as the geodetic precession and was successfully measured by the GP-B experiment.

7. Frame dragging

Let us examine the effect of the gravitomagnetic field caused by the Earth's rotation on the gyroscope. In the GP-B experiment, as illustrated in figure 2, the direction of the gyroscope's angular momentum vector and the gravitomagnetic field are perpendicular. In this case, the Earth's rotation also results in an additional precession of the gyroscopes, whose angular velocity using equation (6.2) is given by

$$\Omega_{FD} = -\bar{b}_R. \tag{7.1}$$

Using equation (5.10) this yields

$$\Omega_{FD} = \frac{GI}{2c^2R^3}\omega_E \tag{7.2}$$

which is identical to the angular velocity derived from general relativity and is referred to in equation (2.2) as the frame dragging effect.

8. Discussion

The relativistic phenomena examined in the GP-B experiment can be explained with the help of gravitomagnetism. The derivation uses a toolbox mathematically analogous to the toolbox of electromagnetism such as the gravitomagnetic law of induction, the effect of a torque of the gravitomagnetic field on the gyroscope. This methodology will hopefully make the explanation of the GP-B experiment more available for physics students.

The gravitomagnetic phenomenon can also be derived from special relativity as shown by Karlsson (2006). The magnitude of the gravitomagnetic force shown by Karlsson is a factor of 2 smaller than the one implied by general relativity. Karlsson's method is suitable

for demonstrating the phenomenon of gravitomagnetism but it is not able to explain the GP-B experiment quantitatively.

One can think of the electromagnetic analogy of the GP-B experiment, as an electrically charged gyroscope (i.e. a magnetic dipole) orbiting another charged insulator, carrying the opposite charge. In this case, it is the Coulomb force that keeps the charged gyroscope in orbit and the effect of gravity can be ignored. Let us take two spherically symmetric insulators with homogeneous mass and charge distribution of masses m and M ($M \gg m$) and of opposite charges q and Q. Let the insulator of mass m be an ideal, torque-free gyroscope with a spin of $\boldsymbol{S}$ orbiting around M on a circular orbit of radius R. The magnetic dipole moment of the gyroscope is $\boldsymbol{m}_{\mathrm{M}} = q\boldsymbol{S}/(2m)$. Because the electromagnetic torque acting on the magnetic dipole is formally a quarter of the gravitomagnetic one, the Thomas precession exactly cancels out the electromagnetic precession caused by the orbiting charged insulator. It means that there is no electromagnetic 'geodetic effect' for gyroscopes with homogeneous mass and charge distribution. However, it is not the case for the spinning electron in a H atom, because the magnetic dipole moment of an electron is coupled with its spin as $\boldsymbol{m}_{\mathrm{Me}} = e\boldsymbol{S}/m_{\mathrm{e}}$, the Thomas precession cancels only one half of the electromagnetic precession. In a Bohr-model approximation the angular velocity of the precession of the electron spin axis is

$$\Omega_{\mathrm{e}} = \frac{ke^2}{2c^2 m_{\mathrm{e}} R}\omega_{\mathrm{e}}, \tag{8.1}$$

where R and ω_{e} are the radius of the orbit and the orbital angular velocity of the electron, respectively.

When the charged central object is rotating around an axis oriented in the orbital plane of the revolving gyroscope, the frame dragging effect of the electromagnetic gyroscope with homogeneous mass and charge distribution exists in this electromagnetic thought experiment. Let the central object of mass M and of moment of inertia I rotate at an angular velocity ω_{C}; then, the magnitude of the electromagnetic 'frame dragging', the angular velocity of the spin axis of the gyroscope of mass m, is

$$\Omega_{\mathrm{EMFR}} = -\frac{kqQI}{8c^2 mMR^3}\omega_{\mathrm{C}}. \tag{8.2}$$

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