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To cite this article: Zoltán Csernovszky 2019 *J. Phys.: Conf. Ser.* **1223** 012004

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From harmonic oscillation to chaotic motion of a compass

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Abstract. With secondary school students at the age of 17-20 years old, we described compass motion in uniform-immobile, uniform rotating and their superposed magnetic fields. As the students have already known the kinematics, dynamics and energy analysis of a simple pendulum in harmonic case, we tried to find analogies between pendulum and compass. Using Dynamic Solver freeware software and two representation types, phase-plane and stroboscopic projection, we simulated and analysed compass motions in more and more complex cases. Finally, we searched chaotic attractors, the sign of chaotic motion, on stroboscopic projections. A new type of representation, a video where every picture is a stroboscopic projection, helped to understand our results. This study was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences.

1. Conservative pendulum and compass motions in uniform immobile fields

The description of a simple pendulum motion and phenomena like harmonic, damped or forced oscillations and chaotic motions play an important role in mechanics [1]. Let a pendulum be an object of mass m , fixed to the end of a massless rigid bar of length ℓ , the other end of the bar is fixed to a support O . Place it in a uniform immobile gravitational field of \vec{g} gravitational acceleration (see Figure1) and move the object from its stable equilibrium position by an angle x and start to follow the conservative motion with its fundamental equation:

$$-m\ell \cdot \vec{g} \cdot \sin x = \theta_p \cdot \beta \quad (1)$$

We find on the left side of (1), the moment of gravitational force, referring to the support O . The negative sign indicates, that the moment of force rotates the pendulum towards the stable equilibrium position, determined by the direction of gravitational field. On the right side, we see the product of moment of inertia ($\theta_p = m \cdot \ell^2$) and angular acceleration of pendulum (β) referring to the support at O .

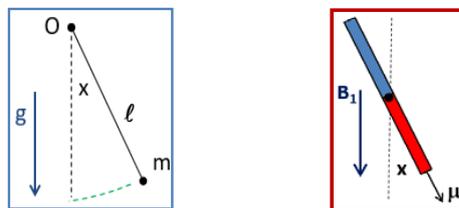


Figure 1 Simple pendulum (left panel, side view) and compass (right panel, from above) in homogenous fields

We can formulate the same type of equation for a compass, placed in a uniform immobile magnetic field of magnetic induction \mathbf{B}_1 the analogous quantities of Table 1:

$$-\mu \cdot \mathbf{B}_1 \cdot \sin x = \theta_c \cdot \beta \quad (2)$$

In (2) μ is magnetic dipole moment, \mathbf{B}_1 is magnetic induction, θ_c is moment of inertia of a compass turning around the vertical axis passing through the centre of the compass. In (2), the angle x , in yellow on Figure 2, left panel, is between the direction of the compass needle and the magnetic induction.

Pendulum	\vec{g}	$m \ell$	θ_p	$E_p = \frac{1}{2} \theta_p \cdot \omega^2 + m \ell \cdot \vec{g} \cdot (1 - \cos x)$ (3)
Compass	B_1	μ	θ_c	$E_c = \frac{1}{2} \theta_c \cdot \omega^2 + \mu \cdot B_1 \cdot (1 - \cos x)$ (4)

Table 1 Pendulum and compass: analogous quantities. E_p and E_c are rotational kinetic and potential energies

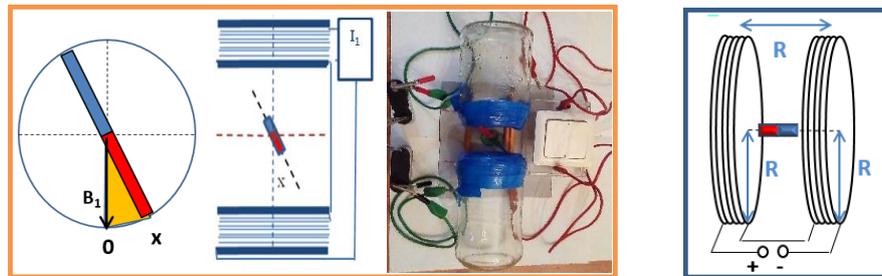


Figure 2 Left panel: Compass motion in uniform magnetic field of a Helmholtz-coil, Right panel: Helmholtz-coil

If we fix the reference level of potential energies to stable equilibrium, we see that the systems are equivalent from the point of view of energy too (Table 1) [2].

To unify (1) and (2), we use a dimensionless fundamental equation, with parameter value b , so:

$$-b \cdot \sin x = \beta \quad (5)$$

For the compass $b = \frac{\mu \cdot B_1}{\theta_c \cdot \Omega^2}$ (6.1) and, for the pendulum $b = \frac{m \ell \cdot g}{\theta_p \cdot \Omega^2}$ (6.2)

2. Helmholtz-coils, uniform immobile, uniform rotating and superposed magnetic fields

To create a uniform immobile magnetic field, take two identical coils of N number of turns on the same axis. Separate the coils by a distance equal to the radius R of the coils. Connect them in series and supply this circuit with a constant direct current of intensity I . The magnetic fields of the coils add, and between the coils, in air of magnetic permeability μ , is a quasi-uniform immobile magnetic field of magnetic induction.

$$B_1 = \frac{8}{5\sqrt{5}} \frac{I \cdot N}{R} \mu \quad (7)$$

This device is called a Helmholtz coil (see Figure 2, right panel). The magnetic field is easily variable. In Figure 2, left panel we show a circuit containing a Helmholtz coil, made from used PC monitor-wire.

Apply in two perpendicular Helmholtz coils (see Figure 3), two identical variable currents, dephased by a quarter of a period. The superposition of the magnetic fields of the Helmholtz-coils results a uniform rotating magnetic field of magnetic induction B_0 which turns at the angular frequency Ω of the currents.

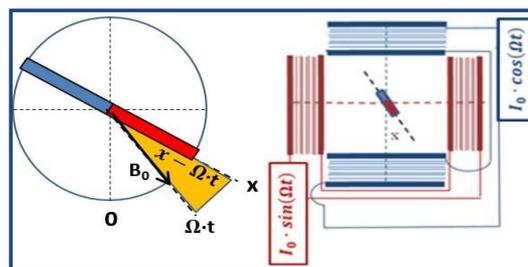


Figure 3 Compass motion in the uniform rotating magnetic field of two perpendicular Helmholtz-coils

To formulate the fundamental equation from a referential frame, use (2) and the yellow angle of Figure 3, which is the difference between actual position of the compass and the direction of the uniform rotating magnetic induction B_0 :

$$-c \cdot \sin(x - \Omega \cdot t) = \beta \quad (8.1)$$

If we fix the referential frame to a rotating field, we can replace the term $(x - \Omega \cdot t)$ by x' and we obtain the same type of fundamental equation as in the uniform immobile field case:

$$-c \cdot \sin x' = \beta' \quad (8.2)$$

We remark that the inertial forces are radial and for the dimensionless c parameter we now have $c = \mu B_0 / \Theta_c \Omega^2$.

The relation
$$x' = x - \Omega \cdot t \quad (9.1)$$

between angles from the rotating and referential frame results in the relation between angular velocities:

$$\omega' = \omega - \Omega \quad (9.2)$$

and implies that angular acceleration is the same, from both frames: $\beta = \beta'$.

If we apply a positive constant direct current in one of the Helmholtz coils, we will have the superposition of a uniform rotating (B_0) and a uniform immobile (B_1) magnetic field (see Figure 4). Using (5) and (8.1), the dimensionless fundamental equation of the compass in the superposed magnetic field will be:

$$-b \cdot \sin x - c \cdot \sin(x - \Omega \cdot t) = \beta \quad (10.1)$$

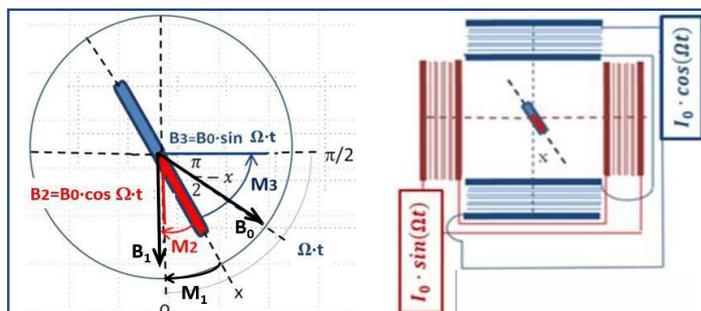


Figure 4 Compass motion in superposed magnetic field, magnetic inductions and moments of force

The energy dissipation, due to friction is considered with a moment $-\gamma \cdot \omega$, proportional to ω , the angular velocity of the compass. The, fundamental equation

$$-a \cdot \omega - b \cdot \sin x - c \cdot \sin(x - \Omega \cdot t) = \beta \quad (10.2)$$

describes dissipative compass motion in the superposed magnetic field, where parameter $a = \gamma / \Theta_c$ measures energy dissipation, b the uniform immobile field intensity, and c the uniform rotating magnetic field intensity [3].

3. Dynamic Solver software and phase-plan representations of uniform immobile and uniform rotating field cases

The realizable experiences are expensive and it is difficult to collect data. For these reasons we decided to simulate compass motion with Dynamic Solver freeware software [4]. After fixing the numerical

method type (Dormand-Prince), independent and dependent variables ($t, x, \omega = dx/dt, \beta = d^2x/dt^2$), initial conditions (x_0, ω_0), parameter values (a, b, c) and time step (Δt), the software resolves the fundamental equation, in every time step. Thus, we will dispose the sequence of angles ($x_1, x_2, x_3 \dots x_i = (t_0 + i \cdot \Delta t)$) and the sequence of angular velocities ($\omega_1, \omega_2, \omega_3 \dots \omega_i$), where $i = 1, 2, 3, \dots$ [5]. Using these sequences, we can choose the representation type for motion analysis.

In Figure 5 we show the representation of the compass motion in a uniform immobile field on the angle-angular-velocity phase plane. Fixing the b parameter at 6.25 (shown as 6,25 in Figure 5), we represent a series of motions with changing initial conditions. Thus, each curve represents one motion, using sequences of angle and angular velocity.

Red and deep red curves represent oscillations (zones ③ and ④).

Green curves separate the blue and red regions (zone ②). In this case the compass has the potential energy of the unstable equilibrium position and goes through the stable equilibrium position with the dimensionless angular velocity $2 \cdot \sqrt{b} = 5$. Here \sqrt{b} is the proper circular frequency and the E/θ dimensionless quotient is 12.5. If initial conditions determine a greater energy, in absolute terms, the compass motion will be rotational around a stable equilibrium point. (See the unclosed blue periodic curves of zone ①.)

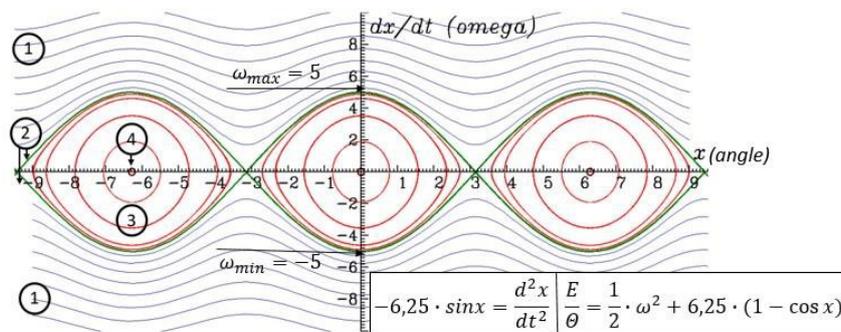


Figure 5 Compass in uniform immobile magnetic field: phase-plan representation, each curve represents one motion

Zone / curve	Motion	E/θ	Initial conditions		
			x_0	ω_0	
①	Periodic	Rotation	Over $2\sqrt{b}$ Under $-2\sqrt{b}$	9 -9	9; 8; 7; 6; 5; 4; 3; 2 -9; -8; -7; -6; -5; -4; -3; -2
②	Closed periodic	Rotation/ oscillation	$\pm 2\sqrt{b}$	$-3\pi; 3\pi$	0
③	Closed periodic	Anharmonic	Oscillation $2\sqrt{b}, -2\sqrt{b}$	$\pm 9; 9 - 2\pi; \frac{\pi}{4}; \frac{\pi}{4} \pm 2\pi; \frac{\pi}{2}; \frac{\pi}{2} \pm 2\pi; \frac{3\pi}{4}; \frac{3\pi}{4} \pm 2\pi$	0
④	Circle	Harmonic		$5 \frac{2\pi}{360}; 5 \frac{2\pi}{360} \pm 2\pi$	0

Table 2 Classification of curves of Figure 5; motion types, energies and initial conditions

If initial conditions determine a smaller energy, in absolute terms, the compass motion will be oscillatory. (See the closed, red periodic curves of zones ③ and ④.) Anharmonic oscillations are represented by ellipses of zone ③. The curves of zone ④ can be considered as circles and represent harmonic oscillations.

For the analysis of the rotating field case compare the phase-plan representations from inertial ($x; \omega$), rotational ($x'; \omega'$), and mixed reference frame ($x'; \omega$). The represented motions are determined by the initial conditions of Table 2. For ($x; \omega$) the representation is equivalent of the immobile case

representation. For $(x'; \omega')$ the only difference will be a vertical shift by Ω , according to (9.2).

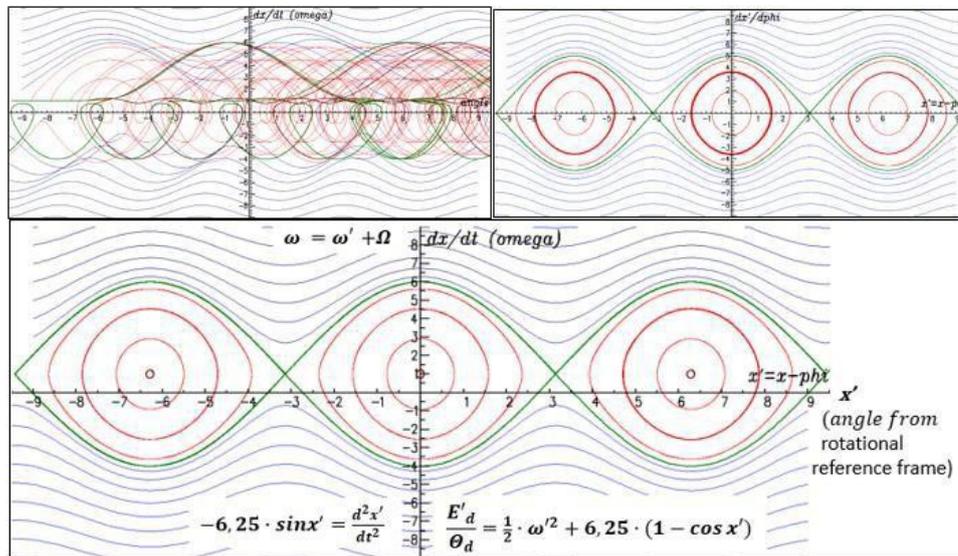


Figure 6 Compass motions in rotational fields from referential, rotational and mixed referential frame

4 Compass motions in superposed fields: stroboscopic projection

The compass motions in the superposed field result in such a complicated phase-space representation, from any reference frame, that it requires simplification. If we consider the period of the rotating field and we introduce the phase of the rotating field $\varphi = \Omega \cdot t$, it is possible to use only the points of equal phase of the sequences of angle- and angular velocities. The representation of the sequence $((0); (0)), ((2\pi); (2\pi)), \dots, (x(i \cdot 2\pi)); \omega(i \cdot 2\pi))$ is described as stroboscopic, since it gives information from the motion at equal time intervals separated by $2\pi/\Omega$. See Figure 7, left panel, and $i = 0, 1, 2, \dots$

Finally, we reduce the representation to the angle interval $[-\pi, +\pi]$. (See Figure 7, right panel.) We will thus display a sequence of points of equal phase on a special phase-plane.

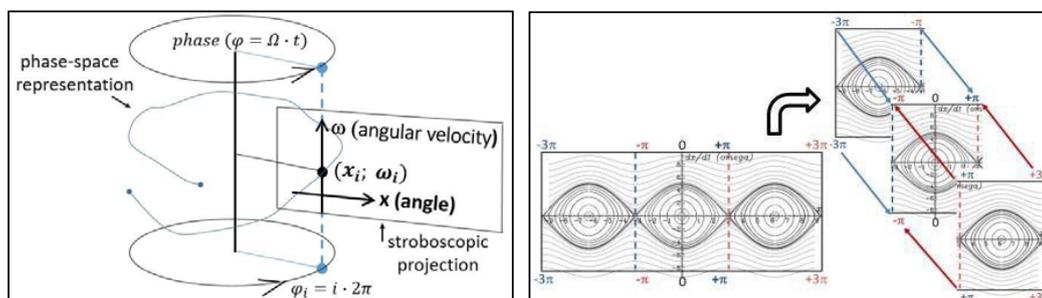


Figure 7 Stroboscopic projection and phase-space. Representation reduction to angle interval $[-\pi, +\pi]$

Figure 8 shows a stroboscopic projection of a conservative compass motion in a superposed magnetic field. We see that this representation contains points and on the horizontal axis we use only the angle interval $[-\pi, +\pi]$. If we fix the b and c parameter values, as in Figure 8, the compass follows unpredictably either the uniform (immobile) or the uniform rotating magnetic field [6]. On enlarged parts of Figure 8 we see oscillations around the rotational (top part) and around the uniform (bottom part) magnetic field. Dimensionless values: $x_0 = -1, \omega_0 = 0, -7.5 < \omega < 15$.

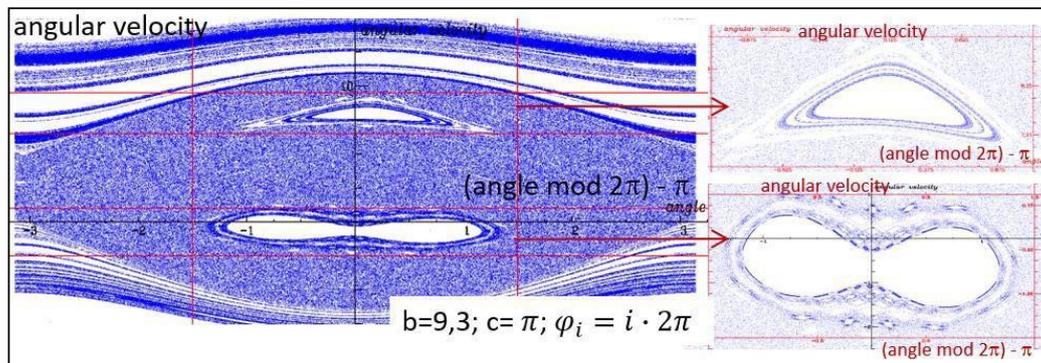


Figure 8 Stroboscopic projection of a conservative compass motion in a superposed field with parameters ($b = 9.3; c = \pi$)

5 Dissipative compass motions in superposed magnetic fields

In the dissipative and superposed magnetic field case, the motion can become chaotic. A chaotic motion is a non-repeating, non-periodic, irregular motion which is to initial conditions. However, chaotic motion in stroboscopic projection is represented by a fractal structure subset, called a chaotic attractor. It attracts the representation of chaotic motion for any initial conditions of this subset, after enough time. [1]

In Figure 9, left panel, we see a chaotic attractor on a stroboscopic projection of a dissipative motion in a superposed field, with $a = 0.1, b = 39$ and $c = 49$. The enlarged figures, from left to right, underline the auto-similar structure of a chaotic attractor.

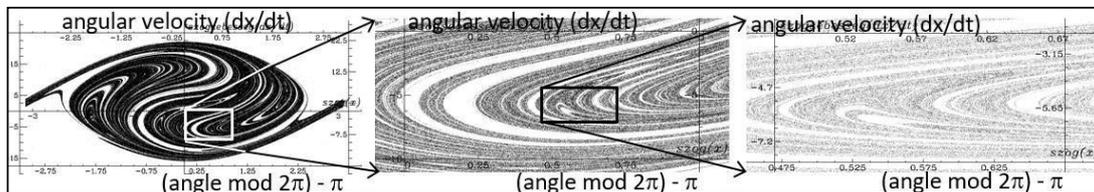


Figure 9 Chaotic attractor on stroboscopic projection (dissipative compass motion in superposed field)

In the stroboscopic projections of Figures 10, 11, 12 we varied a, b, c parameters, fixing initial conditions as $x_0 = 0, 1; \omega_0 = 1$. On the abscissa-axis the angle x is between $-\pi$ and π , on the ordinate-axis the angular velocity ω is between -15 and 20 . All quantities are dimensionless.

To analyse the effect of energy dissipation, we varied the a parameter, fixing b and c parameter-values. We see chaotic attractors on these stroboscopic projections, which indicate chaotic motions at these parameter values. See Figure 10, where $b = 15, c = 15$. Increasing the dissipation, the extension of the chaotic attractor on the phase-plan is reduced.

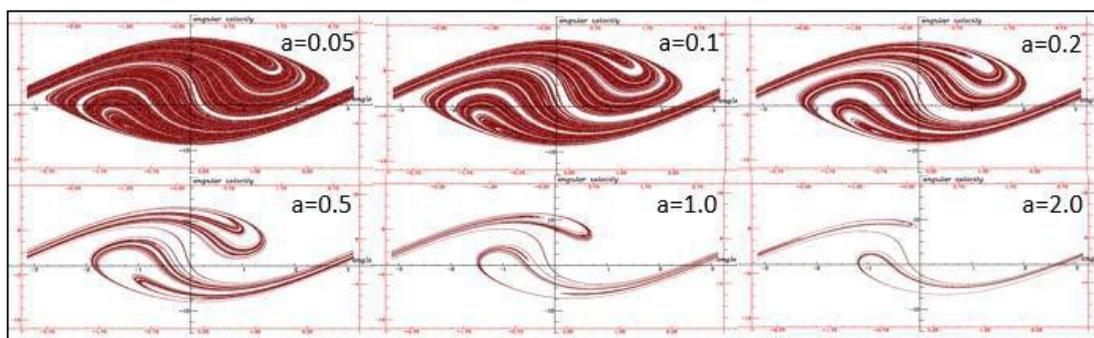


Figure 10 Effect of dissipation on chaotic attractor ($b = 15, c = 15$)

To analyse the effect of the uniform immobile magnetic field, we varied the b parameter value between 0 and 70, fixing dissipation as $a = 0.1$ and uniform rotating field as $c = 15$. In Figure 11, the white intervals ①, ②, ③, ④ indicate the appearance of chaotic attractors.

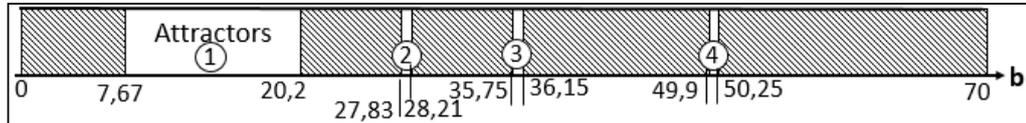


Figure 11 Effect of uniform immobile magnetic field on chaotic attractor appearance ($a = 0.1, 0 < b < 70, c = 15$)

In the interval $0 < b < 7.67$ we haven't got attractors on stroboscopic projections, here the rotating field is dominant. In the interval $7.67 \leq b \leq 50.25$ chaotic attractors appear four times. Figure 12 shows four chaotic attractors of this interval, where $b = 20.8, b = 27.9, b = 36.15$ and $b = 49.93$. The uniform immobile field will be dominant if $50.25 < b$ and chaotic attractors disappeared.

Increasing the uniform immobile field from $b = 20.8$ to $b = 49.93$, we receive the increase of the angular velocity interval from $[-9; 15]$ to $[-15; 17.5]$. See Figure 12.

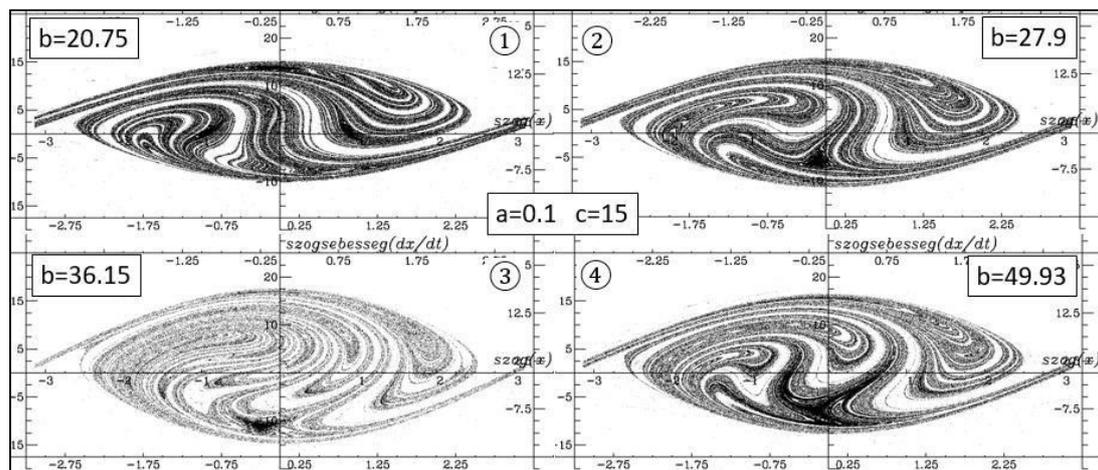


Figure 12 Chaotic attractors and different uniform immobile fields ($a = 0.1$ and $c = 15$)

To analyse the effect of a uniform rotating magnetic field, we varied the c parameter value between 0 and 70, fixing dissipation as $a = 0.1$ and the uniform immobile field as $b = 20$. In the interval $0 < c < 15$ we haven't got attractors on stroboscopic projection. Here the uniform immobile field is dominant. On Figure 13 white intervals ①, ②, ③, ④ indicate the appearance of chaotic attractors. If $58.4 < c$, where the uniform rotating magnetic field is dominant, the attractors disappear.

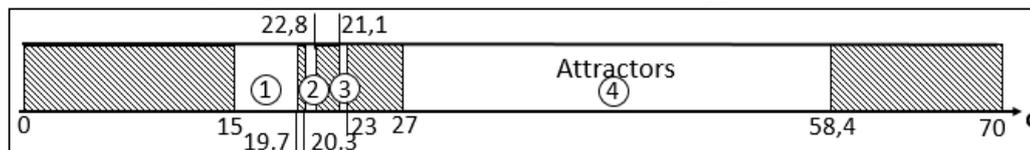


Figure 13 Effect of uniform rotating magnetic field on chaotic attractor appearance ($a = 0.1, b = 20, 0 < c < 70$)

Figure 14 shows four chaotic attractors of this interval, where $c = 15, c = 21, c = 22.9$ and $c = 58.4$. Increasing the uniform rotating field from $c = 15$ to $c = 45$, we receive the increasing of angular velocity interval from $[-9; 15]$ to $[-12.5; 20]$. See Figure 14.

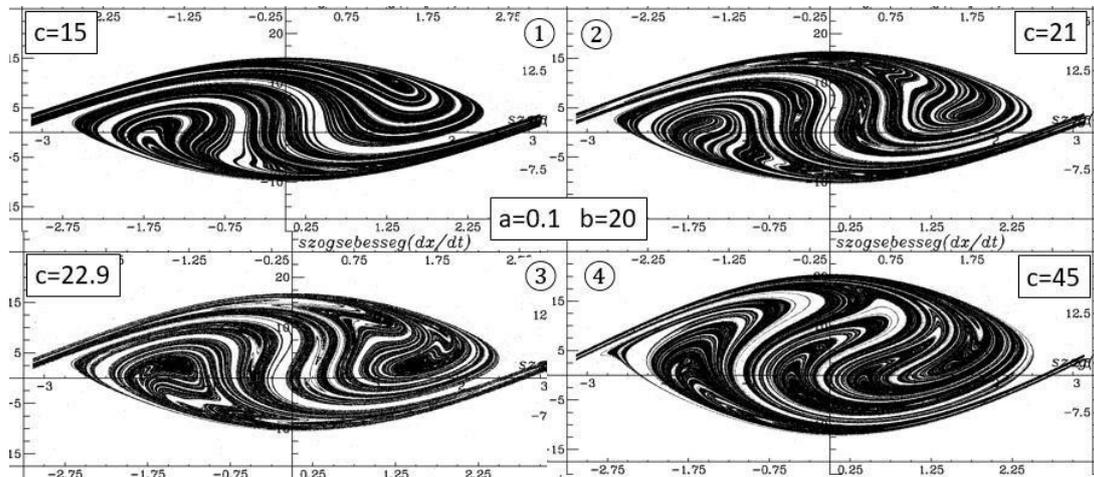


Figure 14 Chaotic attractors and different uniform rotating fields ($a = 0.1$ and $b = 20$)

Realizing many simulations, as in Figures 12, 13 or 14, a video representation is the most useful. In this video every picture can be a stroboscopic projection. In these projections, two parameters (a and b) were fixed, only the third (c) parameter value was changed from 0 to 100 by steps of 1. Finding something special they zoomed in c to have more detailed parameter-map. See a video-detail in Figure 15.

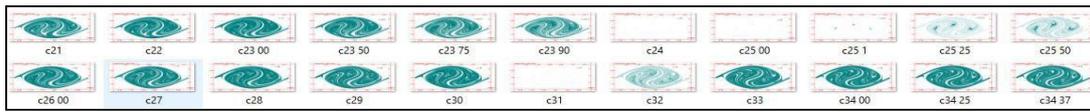


Figure 15 Stroboscopic representations, used in video (detail)

6 Applications in physics education

From the point of view of physics education, the description of a compass motion in uniform immobile, rotating and superposed magnetic fields allows some interesting applications. See *Table 3*.

Current loop Magnetic dipole moment CONSERVATIVE MOTION IN UNIFORM FIELD Helmholtz-coil	Analogy pendulum/compass Motion type-energy CONSERVATIVE MOTION IN ROTATING FIELD Phase plan Rotating reference frame Helmholtz-coils
Dynamic Solver	
Stochasticity CONSERVATIVE MOTION IN SUPERPOSED FIELD Helmholtz-coils	Deterministic chaos Chaotic attractor Fractal structure DISSIPATIVE MOTION IN SUPERPOSED FIELD Stroboscopic projection Different viscosities

Age	Level/Class	Courses/week	Participants	Topics
17-18	Secondary-Upper Sixth Obligatory	Courses + Class activity	2x15	<div style="background-color: #e91e63; color: white; padding: 2px; display: inline-block;">CONSERVATIVE MOTION IN UNIFORM FIELD</div> - Analogies - Pendulum-compass - Realization of a Helmholtz-coil
17-18	Secondary-Upper Sixth Facultative	Project work	15	<div style="background-color: #4caf50; color: white; padding: 2px; display: inline-block;">CONSERVATIVE MOTION IN ROTATING FIELD</div> - Using rotating reference frame - Phase-plane
18-19	Secondary-Facultative Baccalauréat	1 course + Project work	10	<div style="background-color: #2196f3; color: white; padding: 2px; display: inline-block;">CONSERVATIVE MOTION IN SUPERPOSED FIELD</div> - Stroboscopic Projection - Motion analysis
19-	University			<div style="background-color: #ffc107; color: white; padding: 2px; display: inline-block;">DAMPED MOTION IN SUPERPOSED FIELD</div> - Chaotic motion - Chaotic attractor

Table 3 Above: proposed topics and activities, Below: proposed age, number of courses and participants

In the uniform immobile case we introduced analogies between the simple pendulum motion and compass motion. The moment of magnetic force exercised on a current loop placed in a uniform immobile magnetic field, and the equivalence of the magnetic field of a current loop and the magnetic bar are the bases of the introduction of magnetic dipole moment and pendulum-compass analogy [2]. See Figure 16. In physics education the analogical thinking helps to understand explications, descriptions in new situation, because it is based on already-known notions [8].

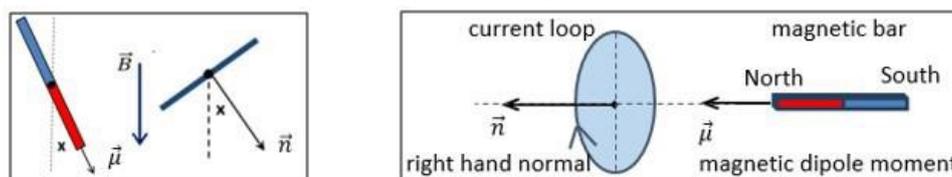


Figure 16 Moment of force in uniform immobile magnetic fields. Magnetic field equivalence of current loop and magnetic bar.

As a class activity we constructed Helmholtz-coils with used PC monitor wires and the compass oscillations was analysed in its uniform immobile magnetic field.

We presented phase-plan representations to associate motion type and energy in conservative cases using the conservation of energy. This type of representation develops abstract thinking, demonstrates the exceptionality of harmonic oscillations and will be a useful method in other domains of physics [4]. The representation of compass motion in a uniform rotating magnetic field on a phase-plan is an elegant example of changing referential frame and using a rotating referential frame to simplify the description.

The analysis of compass motion in superposed magnetic fields requires stroboscopic projection and seeking the conditions of stochasticity is an exciting part of simulation research [6].

Applying chaotic motions in physics education started in mechanics and most of the examples are related to a pendulum. Here we presented a less known, electromagnetic case, the dissipative compass motion in superposed magnetic field. To seek chaotic attractors the students realized many simulations and proposed a new type of representation. To start the analysis, they fixed two parameter values and made stroboscopic projections at every whole number of the third parameter value. But if they found chaotic attractors, they focused on this value, and changed parameter-scale to realize other projections. Finally, they made a video, where every picture was a stroboscopic projection and they summarized the results on a number line, as we described in Section 5.

The notions of deterministic chaos, chaotic attractor and fractal appear in everyday life [8] [9]. The analysis and characterisation of chaotic attractors or the route to deterministic chaos are fruitful chapters of physics education.

ACKNOWLEDGEMENT

This study was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences. Special thanks to Bálint Barcs for cooperation on the topic of video-representation.

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