

# Investigation of chaos in the absence of programming skill<sup>1</sup>

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**Abstract:** Studying chaotic systems at all levels of physics education can highly motivate students to learn physics. In this paper two simple systems, the Duffing oscillator and compass needle motion, are presented, which were used in a high school and in a university setting for teaching the basic ideas of chaotic motion. Studying both systems can begin with their simple mechanical models, which connect the mathematical description to tangible reality for the students. To surmount the mathematical difficulties of the theoretical treatment of the models, freeware software, the Dynamics Solver (DS), was applied. It will be shown how DS can facilitate students' first steps toward chaotic systems both at college and high school levels. Downloadable materials in ZIP format, related to this paper, can be found in our web page ([http://csodafizika.hu/ds\\_girepmptl](http://csodafizika.hu/ds_girepmptl)).

**Key words:** physics education, chaotic problems, Dynamics Solver, Duffing oscillator, nonlinear oscillators.

**Résumé :** L'étude des systèmes chaotiques à tous les niveaux de la formation en physique peut s'avérer très motivante pour étudier la physique. Nous présentons ici deux systèmes simples, l'oscillateur de Duffing et le mouvement de l'aiguille d'un compas, qui furent utilisés dans les cadres pré-universitaire et universitaire pour enseigner les notions de base du mouvement chaotique. L'étude des deux systèmes peut commencer avec des modèles mécaniques simples, qui connectent la description mathématique à une réalité tangible pour les étudiants. Afin de surmonter les difficultés du traitement théorique des modèles, nous utilisons le logiciel libre Dynamic Solver (DS). Nous allons montrer comment DS peut faciliter les premiers pas des étudiants vers les systèmes chaotiques aux niveaux pré-universitaire et universitaire. Le matériel télé-déchargeable en format ZIP et relié à cet article peut être trouvé sur notre page web ([http://csodafizika.hu/ds\\_girepmptl](http://csodafizika.hu/ds_girepmptl)). [Traduit par la Rédaction]

**Mots-clés :** enseignement de la physique, systèmes chaotiques, Dynamic Solver, oscillateur de Duffing, oscillateurs non linéaires.

## 1. Introduction

A dramatic decrease in the number of physics students has been experienced at universities in Hungary, therefore, university and high school teachers have begun to seek ways of teaching science that are more interesting and attractive. Seeking to engage students, the role of the students' motivation has received increased attention. Glynn et al. [1] developed a Science Motivation Questionnaire based on which the dimensions of the students' motivation can be revealed and that can also be used as a tool to evaluate the effectiveness of instructional strategies and materials designed to increase students' motivation. Further investigation of Glynn and Koballa [2] showed that one of the most important components of motivation is intrinsic motivation, which involves learning science for its own sake.

According to our experiences, students' intrinsic motivation can be improved by studying interesting natural phenomena and also with demonstration and self-made experiments. In addition to these, new topics where students can do their own exploratory work can also arouse their interest. In our choice of new topics we have taken into account that one of the main goals of physics education is demonstration of the power of the qualitative description of natural phenomena. This led us directly to chaotic motion. The exploration of chaotic systems is closely connected to the development of computer simulation experimental physics, which yields a possibility to obtain relevant quantitative information about systems that previously could not be studied or could

only be described qualitatively [3, 4]. Besides that, the investigation of chaotic systems can motivate students to understand physics more deeply at all levels of physics education, and their introduction to curricula can provide a new point of view in understanding the relations between deterministic and probabilistic laws. Experiments with simulations can be applied successfully at the introductory level of physics education because we are convinced that students would be able to concentrate much better on understanding the concepts of chaotic motion if they performed stand-alone simulations.

In the following, mechanical models of two forced and damped vibrating systems, the Duffing oscillator and a compass needle motion, will be presented. Descriptions of both models are presented by building the equations of motion step by step. Both at the high school level and in the introductory college courses, students are familiar with Newton's laws and their simple application but teaching mostly focuses on simple problems that can be solved exactly and the solutions of which can be expressed explicitly with mathematical formulas. However, it is worth dealing with problems that should be solved numerically whose solutions can be given only graphically or in tables. The solution of this type of problem can exhibit chaotic behaviour even in simple, but nonlinear, systems.

## 2. The Dynamics Solver (DS) program

Working with computers generally requires advanced programming skills that may discourage students; even if they at-

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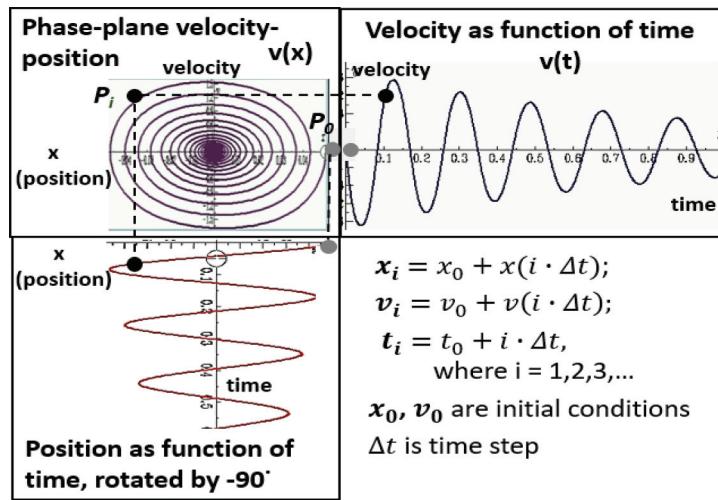
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**Fig. 1.** Phase-plane and time-dependent representations. [Colour online.]



tempt complicated manipulations it may distract them from the conceptual basis of the topic. The present paper has a twofold goal, on the one hand we would like to demonstrate that students at an introductory level of physics can more easily understand the basic concepts of chaotic motion if they study the motion of real mechanical systems; on the other hand, we suggest the DS program as a didactic tool for surmounting the mathematical difficulties of the simulation of motions. The DS is a freeware software that was developed for the simulation of dynamic systems [5, 6]. Its use does not require any programming skill at all. Authors of the present paper have employed it previously with success in their research on chaotic systems [7–10]. In this paper the educational aspect of the use of DS is emphasized. It will be shown how students can be helped to become familiar with intricate concepts of chaos theory by using the DS [5].

DS is a versatile professional program that was developed to study differential equations, (continuous and discrete) nonlinear dynamical systems, deterministic chaos, mechanics, etc. For instance, you can draw phase space portraits, Poincaré maps, calculate Lyapunov exponents, provide bifurcation diagrams, attraction basins, etc. [6]. However, for beginners, it is enough to know that the program can solve systems of differential equations if they are entered in the form of autonomous differential equations.

In the following we present how DS was used as a teaching aid in a B.Sc. course of the John von Neumann University, Kecskemét, Hungary, and with 17–19 year old students as a class activity in Berzsenyi High School, Budapest, Hungary. The students could study the real behaviour of the oscillators with DS simulations. By changing the parameter values, students can map the various types of motion of the oscillator. The fractal character of the strange attractor can be convincingly experienced on representations generated by DS.

## 2.1. Didactic challenges

When we are planning new teaching material for high school students (or for introductory college physics courses), we should first look carefully through the background material of the topic chosen considering both the necessary physical and mathematical concepts using relevant reference books. For chaotic motion, the book by Tél and Gruiz [11] was used as reference, while for physics we used Halliday et al. [12] and Young and Freedman [13]. Thereafter an educational breakdown of the scientific material should be constructed. This entails a “translation” of the physical and mathematical concepts into a language understandable to the students. This should occur in iterative steps where the teach-

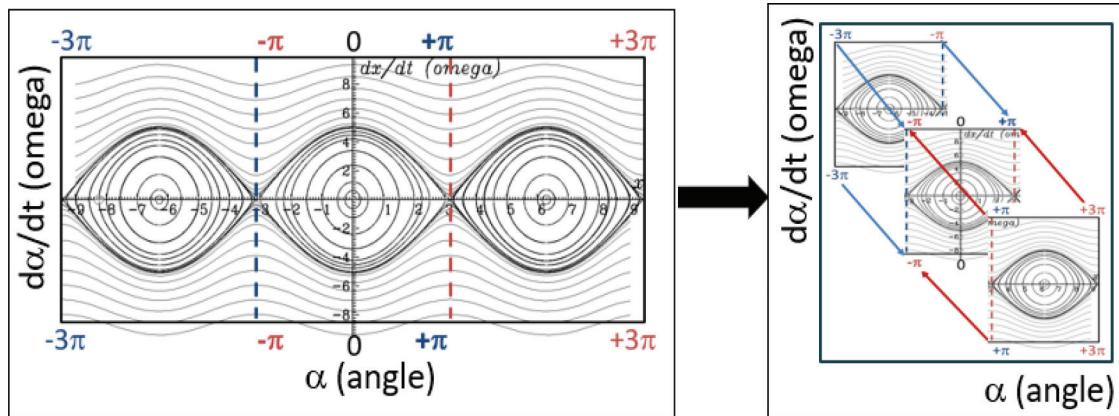
ing materials are checked and modified based on experiences of classroom work [14].

In our case the most difficult task is the treatment of the mathematics; students should be familiar with calculus, the concept of phase space, and at least the Euler method for the solution of differential equations. Chaotic systems can generally be described by an autonomous system of differential equations. These systems consist of first-order ordinary differential equations where the derivatives of the variables are expressed explicitly. These equations are the input of the DS. At this point it is worth considering what kind of information is needed to calculate the change over time in a mechanical system. Take a body of mass  $m$  moving along a straight line as an example. If the coordinate  $x(t)$  and velocity  $v(t)$  of the body are known at the moment  $t$ , its consecutive coordinate after the lapse of an “appropriately” small  $\Delta t$  time can be calculated in the form  $x(t + \Delta t) = x(t) + v(t)\Delta t$  (the calculation will be more accurate the smaller  $\Delta t$  is). The velocity can be obtained similarly with the equation  $v(t + \Delta t) = v(t) + a(t)\Delta t$  if the acceleration  $a(t)$  of the body, at moment  $t$ , is known. Students should understand that numerical solutions of the differential equations are based on this simple method, but in DS more developed procedures are applied that provide better results. In this respect the DS can be considered a black box. If we write the input data in a suitable form it provides the results, for example, the solution of the equations of motion of the system investigated in graphical or tabular form.

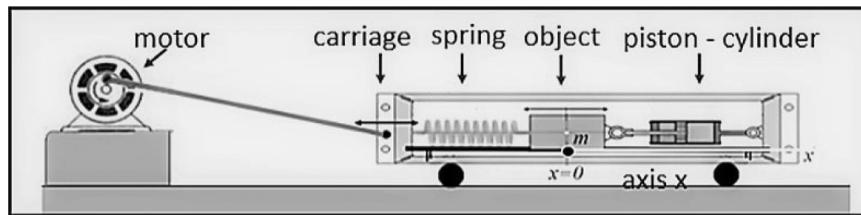
One of the most demonstrative tools for representing the solution of the equations of motion is the phase-space. The DS effectively helps to understand the manipulations with phase space. As a first step we apply, then we present, a phase-plane where one variable of the system is represented as function of another one; see Fig. 1. This representation is the basis of the investigation of chaotic systems. Though it is no more difficult than time-dependent representations, students find it more intricate and hardly understandable; therefore, we suggest some activities to strengthen its conceptual basis and to illustrate it in use.

Computers generally use dimensionless quantities (numbers) so DS can also be applied effectively if the equations are given in dimensionless form. Although the construction of the dimensionless equations seems to be a simple task for an expert, according to our experiences this procedure is very strange and difficult for students. Therefore, in the DS programs, which were made for students, we find examples with dimensional parameters, and in the high school tutorials where dimensionless equations were

**Fig. 2.** Stroboscopic projection: reduction of the projection to interval  $[-\pi, \pi]$ . [Colour online.]



**Fig. 3.** Schematic representation of the mechanical system investigated.



applied, the elimination of the dimensions was presented by the teacher.

The representation of a periodically forced system can result in conceptually complicated phase-space because, to take into account the forcing, the number of variables in the system has to increase by one. We watch the motion only at the same phase as the driving force, that is, snapshots are plotted. It looks as if the photograph of the moving body were made by periodic (stroboscopic) flashes. Therefore, this projection is called stroboscopic. Finally, as the driving force is  $2\pi$  periodic, we reduce the projection to the interval  $[-\pi, \pi]$  (Fig. 2).

### 3. The Duffing oscillator

#### 3.1. Mechanical model

The Duffing oscillator can be modelled by the following mechanical system, which can be discussed in possession of a relatively simple basic level knowledge of physics: the motion of a body oscillating along a straight line with a mass of  $m$  will be studied. See the outline of the mechanical system in Fig. 3.

A carriage rolling easily on a horizontal rail can be pushed to-and-fro using an eccentric shaft. Inside it, a block of mass  $m$  can glide without friction on a track mounted on the carriage. The left end of the block is fixed to the front wall of the carriage with an unstretched spring, and the other end of it is fixed to the rear wall through a piston moving viscously in a cylinder.

It is clear that the centre of mass of the body will move rectilinearly in the proximity of the  $x = 0$  point, and this is the motion we intend to describe. The relevant state variables of the system are the displacement  $x(t)$  and the velocity  $v(t)$  of the body. For students familiar with Newton's laws it is clear that the problem should be solved by the use of Newton's second law, according to which the acceleration of a body is proportional to the net force acting on the body:  $ma = \Sigma F$ , which can be rewritten in the form  $\ddot{v} = a = (1/m)\Sigma F$ .

However, the mathematical description of the forces is not simple. For example, when the eccentric shaft moves the cart, the positions of both ends of the spring are changing, and the damp-

ing force depends on the velocity of the mass relative to the cart. We can see later that both, but especially the latter, facts dictate to fix the coordinate system to the cart, because this simplifies the formula of the spring and damping forces. So, let the line of the track be the  $x$  axis of the coordinate system, the origin of which is at the position of the centre of mass when the spring is relaxed.

Remarks: The choice of the non-inertial frame can cause conceptual difficulties for high school students, but the treatment of the problem in the laboratory system is also very complicated due to the geometry of the spring elongation and the description of the damping force. Therefore, it seems to us that it is worth avoiding those troubles by using the frame of the moving cart.

Here we use the usual symbol for the derivative, of course for beginners the meaning of the notation has to be explained more cautiously.

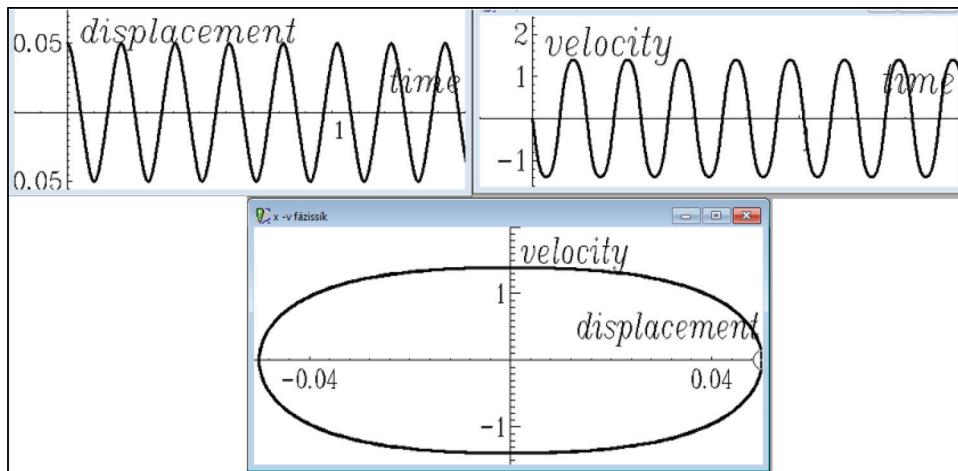
#### 3.2. Mathematical model

The general mathematical (dynamic) model of the body oscillating in the system depicted in Fig. 3 will be built up in this section in four steps, which allows students to be gradually introduced to the use of DS. During the solution, the forces are taken into account gradually and the net force become more and more complex. The steps relating to the forces are

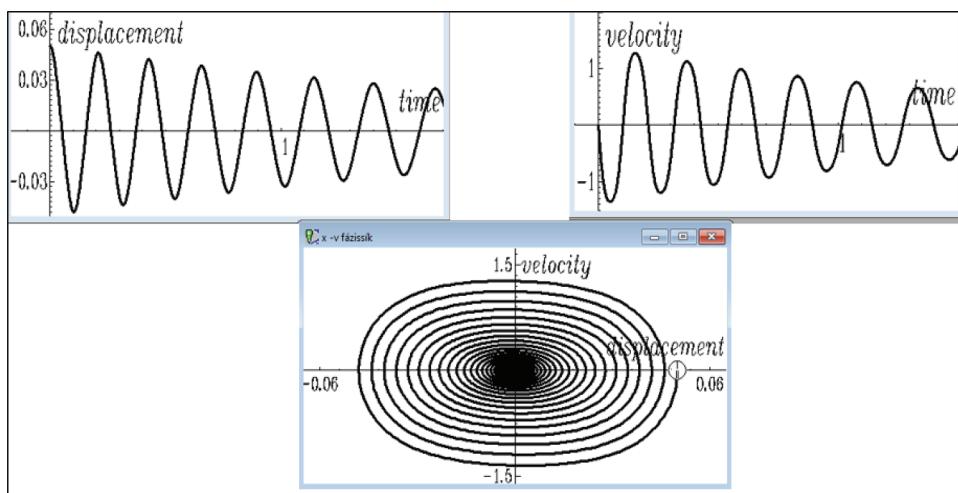
1. linear spring force;
2. linear spring force and viscous damping;
3. changing the linear force to a nonlinear one; and
4. a periodic external driving force (inertial force) exerted on the body.

During the classroom work the equations must be written in the proper form for DS, generally in the following form: the rate of change of  $x$  over time,  $dx/dt = v$ ; and the rate of change of  $v$  over time,  $dv/dt = F_{\text{net}}/m$ . In the first three steps, the eccentric shaft does not move the carriage, so it can be regarded to be fixed to the rail. This means that the reference frame is an inertial one. The peri-

**Fig. 4.** Simple harmonic motion taken from DS ( $m = 0.5$ ,  $D = 200$ ,  $k = 0$ , and  $E = 1.5 \times 10^5$ ). [Colour online.]



**Fig. 5.** Damped oscillation taken from DS ( $m = 0.5$ ,  $D = 200$ ,  $k = 0.5$ , and  $E = 1.5 \times 10^5$ ). [Colour online.]



odic forcing is switched on only in the fourth step when the complete solution is given.

During the classroom work the equations referring to the consecutive steps of the solution have to be discussed in detail. However, for the sake of brevity here this discussion will be summarized. The first equation is, in every step, the rate of  $x$  over time,  $dx/dt = v$ , while the second step is to give the rate of change of the velocity. The net force in the first step is a linear spring force,  $-Dx$ ; in the second step, the linear spring force is supplemented by a viscous damping force,  $-Dx - kv$ . In the third step the nonlinearity is brought into the spring force supposing that the spring is hardening or softening  $-Dx - Ex^3 - kv$ .

In each step it is recommended to run the appropriate DS program (in the first and second steps the *harmonic\_oscillator.ds* and in the third step the *anharmonic\_oscillator.ds* program). Figures 4–6 show the plots of solutions obtained for the consecutive steps, respectively. The upper parts of the figures show the displacement-time functions, while in the lower parts the trajectories in the displacement-velocity phase plane can be seen.

Figure 4 shows a linear harmonic oscillation, the trajectory of which is a closed curve; this is called *limit cycle*. In Fig. 5, a damped oscillation appears. The displacement-time function is obviously a sinusoidal oscillation with, at first sight, an exponentially decreasing amplitude, while the trajectory in the displacement-

velocity phase plane is a spiral that leads into a *stable state* called a *fixed point*, in this case, to the origin, where the body remains at rest. The most interesting result shown in Fig. 6 is that the frequency of the damping oscillation is decreasing with time.

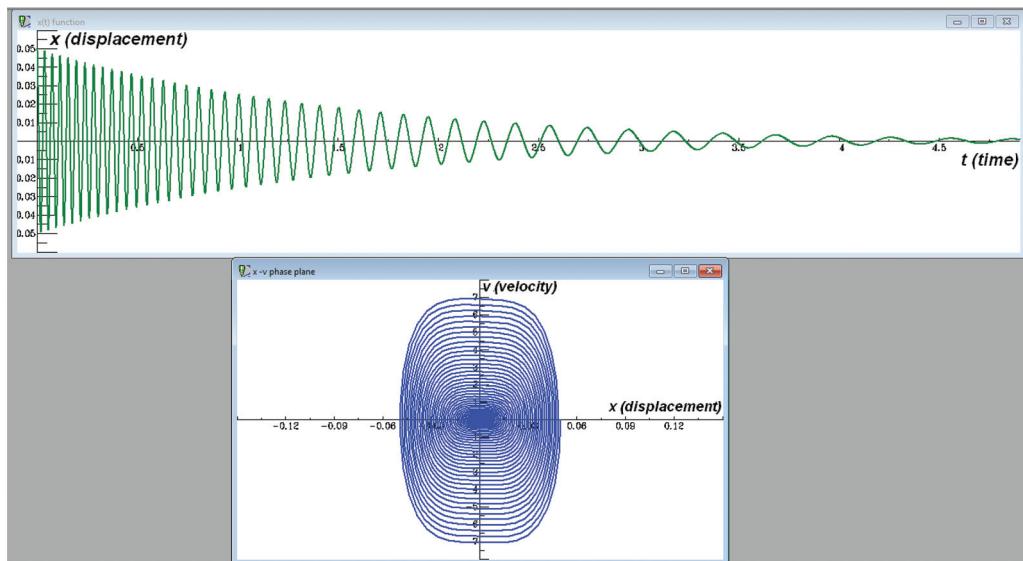
In the last step the driving is activated, and it sets the carriage into periodic motion. This means that the reference frame of the accelerating carriage becomes a non-inertial one, therefore an inertial force  $-ma_C$  is exerted on the body of mass  $m$ , where  $a_C$  is the acceleration of the carriage. If the motion of the carriage takes place according to the function  $x_C(t) = A_0 \cos[(2\pi/T)t]$ , where  $x_C(t)$  is the displacement-time function of the carriage,  $A_0$  and  $T$  are the amplitude and the period, consequently the acceleration is

$$a_C(t) = -\left(\frac{2\pi}{T}\right)^2 A_0 \cos\left(\frac{2\pi}{T}t\right)$$

Because of this, the inertial force,  $F_i(t) = -ma_C(t) = m(2\pi/T)^2 A_0 \cos[(2\pi/T)t]$ , acting on the moving body appears as a periodic driving force in the final form of the equations of motion. Introducing the notation  $F_0 = m(2\pi/T)^2 A_0$ , the acceleration of the carriage will be

$$a_C(t) = -\frac{D}{m}x - \frac{E}{m}x^3 - \frac{k}{m}v + \frac{F_0}{m} \cos\left(\frac{2\pi}{T}t\right)$$

**Fig. 6.** Image of an anharmonic damped oscillation taken from DS ( $m = 0.5$ ,  $D = 200$ ,  $k = 1$ , and  $E = 8 \times 10^6$ ). [Colour online.]



At this point, however, one must reconsider the number of variables needed to specify the complete state of the system. From a physical perspective it is clear that the displacement  $x(t)$  and the velocity  $v(t)$  are no longer enough to determine the values  $x(t+dt)$  and  $v(t+dt)$  of these quantities in the next step of the iteration, because they are obviously influenced by the dynamic effect of the external force acting at the moment  $t$ , which is characterised by the phase  $\varphi = (2\pi/T)t$  of the periodic driving force.

This is then, how the phase-space of the forced oscillation will be expanded to three dimensions: *besides displacement  $x$  and velocity  $v$  the phase  $\varphi$  of periodic forcing will be introduced as a new variable*. The phase is a linear function of time, consequently its rate of change (i.e., the derivate) is constant:  $d\varphi/dt = 2\pi/T$ . When step 1 is completed with step 3, the differential equations describing the dynamics of the periodically forced damped anharmonic oscillation, in other words of the Duffing oscillator, can be obtained

$$\frac{dx}{dt} = v \quad \frac{dv}{dt} = -\frac{D}{m}x - \frac{E}{m}x^3 - \frac{k}{m}v + \frac{F_0}{m}\cos\varphi \quad \frac{d\varphi}{dt} = \frac{2\pi}{T}$$

Let us run the *Duffing-oscillator.ds* program. The respective values of the selected physical parameters (specified in SI units) may be as follows (for the order of magnitude of the range of parameters, see for instance refs. 1 and 2):

$$\begin{aligned} m &= 0.5 \text{ kg} & D &= 2 \times 10^2 \text{ kg/s}^2 & E &= 8 \times 10^6 \text{ kg/m}^2\text{s}^2 \\ k &= 1 \text{ kg/s} & F_0 &= 50 \text{ kg m/s}^2 & \text{and } T &= 0.3 \text{ s} \end{aligned}$$

In Fig. 7, the lower graphic window displays the  $v(t)$  velocity-time function and it can be clearly seen that it does not exhibit any regular behavior: the *chaotic nature appears*. In the upper graphic window, the stroboscopic map of  $x-v$  (displacement-velocity) phase plane is displayed (as if a series of photographs were taken using a stroboscope lamp's light signals flashing with a cycle time  $T$ ; to better understand how it works, see the video *cjp-2019-0466suppl.mp4*<sup>2</sup>). The strange shape plotted in the  $x-v$  phase plane using the stroboscopic imaging technique is called the

chaotic attractor of the Duffing oscillator, showing an interesting geometric structure, bearing the characteristics of a fractal pattern.

### 3.3. General features of chaotic systems

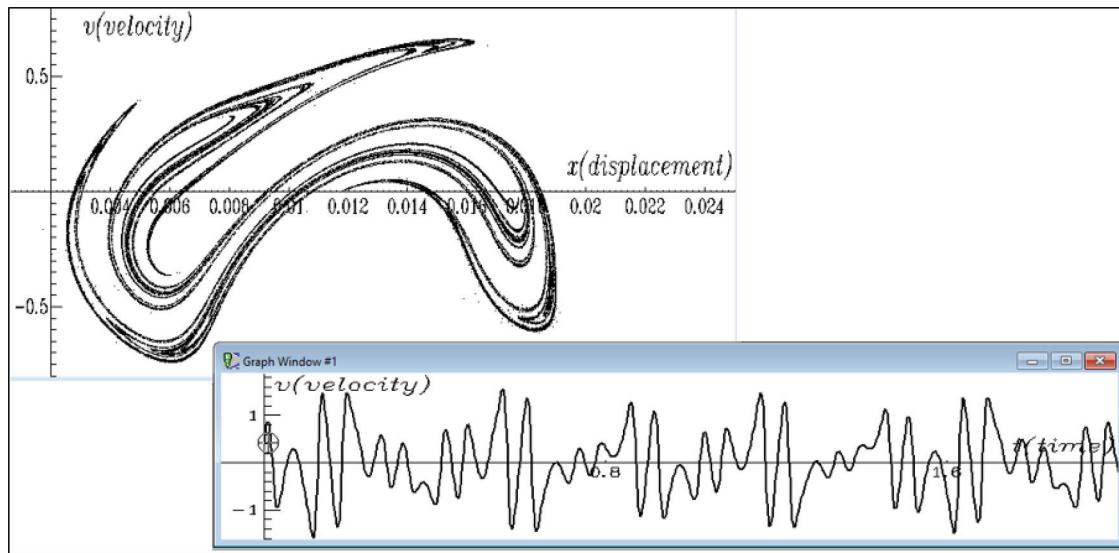
With the help of DS, an additional key feature of chaos can be studied as well in a simple way, this is the *extreme sensitivity* of the behaviour of a chaotic system to the *initial conditions*. This means that very different time dependence can be experienced even if the system is started from very close initial conditions with the same values of the other parameters, in other words, the customary approach: “little initial error resulting in a little deviation of the result” will not be met in this case. The most spectacular way of illustrating this extreme sensitivity is to plot the so-called *ensemble diagram*. This diagram depicts the change over time of a variable in the chaotic system starting from multiple initial conditions very close to each other. Let us run the *Duffing\_oscillator\_ensemble.ds* problem file. When the parameter values defined above are entered into the programme, it shows the velocity function  $v(t)$  in five different, but very close initial displacement  $x_0$  values (in order with actual values of  $-0.0006, -0.0003, 0, +0.0003, +0.0006$ ), while the initial value of the other two variables of the system is always identical:  $v_0 = 0$  and  $\varphi_0 = 0$ . The plot of the velocity functions is shown in Fig. 8. It is clearly seen that the velocity curves run in parallel for a while, then diverge entirely. In other words, the extremely small differences in the initial values result in entirely different behaviour after the lapse of a characteristic time period.

Based on the results obtained from the investigation of the Duffing oscillator, the general features of the chaotic motion occurring in simple systems can be deduced as follows: Simple systems are those that can be described with a few variables (low degree of freedom) and with deterministic rules. The motion exhibits

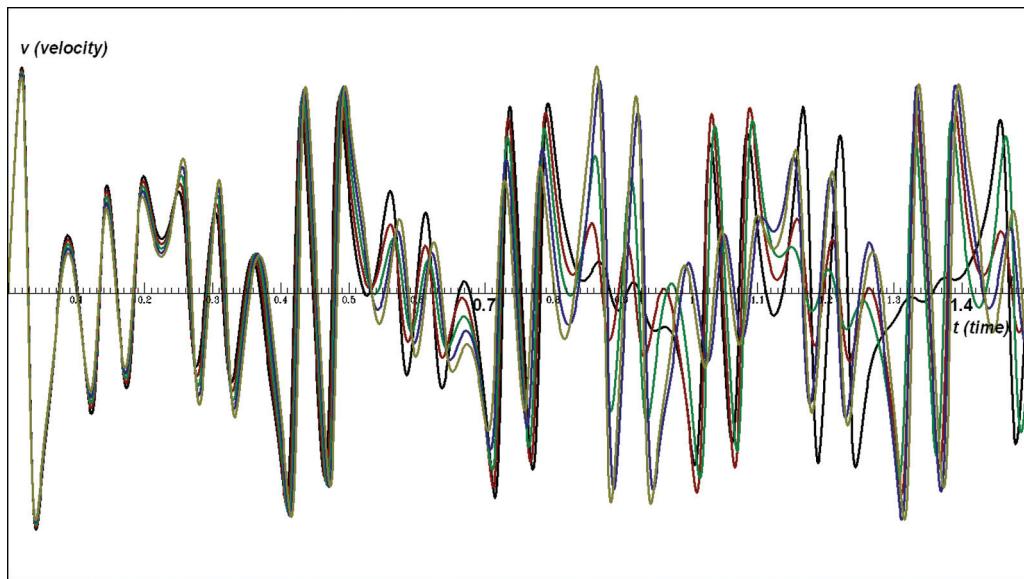
- irregular (non-periodic, complicated) behaviour;
- long-term behaviour in the phase space that is characterised by a strange geometric structure, the attractors in the phase space show fractal geometry;

<sup>2</sup>Supplementary data are available with the article through the journal Web site at <http://nrcresearchpress.com/doi/suppl/10.1139/cjp-2019-0466>.

**Fig. 7.** Chaotic behaviour of the Duffing oscillator taken from DS. [Colour online.]



**Fig. 8.** Ensemble diagram of the Duffing oscillator in DS. [Colour online.]



- extreme sensitivity to the initial conditions, consequently, prediction of the long-term behaviour is practically impossible, concerning this only probabilistic forecasts can be given).

#### 4. Compass needle

Here we present a description of the motion of a compass needle that is under the influence of a uniform static and a uniform rotating magnetic field. It was used as a teaching aid for 17–19 year old students, as a class activity at Berzsenyi (2019) and Kölcsény (2017) high schools, Budapest, Hungary (BHS and KHS, respectively). At KHS, in obligatory physics courses, we presented the analogy of a pendulum and compass needle to apply the well-known mechanical description of harmonic oscillation of a compass needle (4–4 courses with four groups of 17 students). We hoped to motivate students to choose mandatory physics courses, where the DS and the phase-plane were presented for 17 students, in four courses. The next school year we continued the project with 10 students (four courses) to show stroboscopic projection

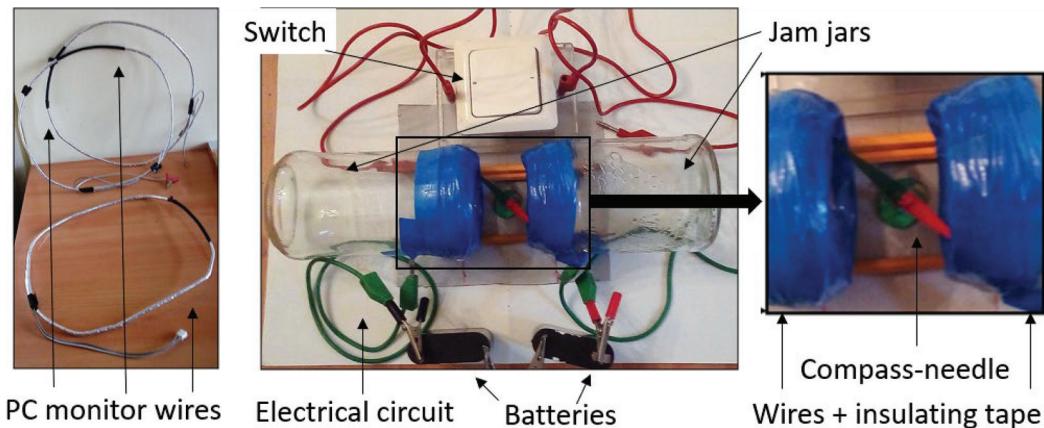
and search chaotic attractors. Finally, “stroboscopic videos” were created by three students to summarize our results (four courses). At BHS, we worked with 10 students of maths and physics specialized classes. We focused on dimensionless equations and stroboscopic projection of conservative oscillators (eight mandatory courses).

The behaviour of the compass needle can be described with a differential equation that is analogous to that derived from the mechanical model of the Duffing oscillator, which will be exploited in the following description.

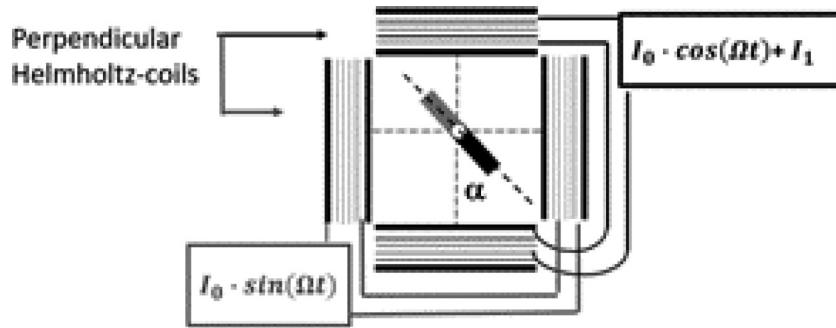
#### 4.1. Experimental setup

A pair of Helmholtz coils consists of two identical circular coils of the same axis, and the distance between the coils is equal to their radius. If the coils are supplied with the same direct current, circulating in the same direction in both coils, they produce a quasi-uniform magnetic field in the region between them. In our high school project, we built Helmholtz coils from used PC mon-

**Fig. 9.** Compass needle in the magnetic field of a “monitor wire – jam jar” Helmholtz coil. [Colour online.]



**Fig. 10.** Compass needle in the superposition of magnetic fields of two pairs of Helmholtz coils (top view).



itor wires, jam jars, and insulating tape (Fig. 9). The coils were supplied by stable and well controllable direct current, which made it possible to vary the intensity of magnetic field easily. Placing a compass needle between the coils, in the center of the symmetry axis, we were able to realize harmonic and damped oscillations and compare it with the results obtained by DS simulations.

If two identical pairs of Helmholtz coils, the axes of which are perpendicular, are supplied by equal alternating electric current dephasing by a quarter of a period, the superposition of their magnetic fields results in a rotating magnetic field, which rotates by the angular frequency of the currents. To have a damped, driven, nonlinear electromagnetic oscillator, place a compass needle in the section of the axis of the coils and then apply an additional constant direct current in one of the Helmholtz pairs (Fig. 10).

Building a device to produce a uniform rotating magnetic field was beyond our technical possibility, so the motion of the compass needle was described only by DS simulations, using phase-plane representations. The DS was presented at optional courses for interested students who had the option to complete project works as well. For didactic reasons, more and more complicated magnetic fields were supposed to act on the compass needle. However, the final goal was the study of chaotic motion and to introduce attractors (see Table 1).

#### 4.2. Fundamental equations

The equations of motion can be written according to the rotational analogy of Newton's second law [13], which states that the net torque  $\Sigma M$  acting on a body, equals the product of the body's moment of inertia  $\Theta$  and angular acceleration  $\beta$ : ( $\Sigma M = \Theta\beta$ ). In general the following torques are exerted on the needle:

1. The restoring torque of the uniform field of the Helmholtz pair:  $-\mu B_1 \sin\alpha$ , where  $\mu$  is the magnetic moment of the needle,  $B_1$  is the magnitude of the uniform magnetic field, and  $\alpha$  is the angular displacement of the needle, measured from the stable equilibrium position of the needle. For small angles this torque can be approximated with  $-\mu B_1 \alpha$ .
2. A driving torque due to the rotating magnetic field  $-\mu B_0 \sin(\alpha - \Omega t)$ , where  $B_0$  and  $\Omega$  are the amplitude and the angular frequency of the rotating field, respectively.
3. A viscous damping force:  $-\gamma\omega$ , where  $\omega$  is the angular velocity and  $\gamma$  is the viscosity of the fluid at the axle of the compass needle.

The general equation of the driven, damped, nonlinear oscillator is built in step by step, as was done in the case of the Duffing oscillator, and an activity was organized around each step for the students. The steps are given in Table 2.

#### 4.3. Preparations for simulation

First, the basic second-order differential equation has to be written in the form of a system of first-order differential equations, introducing angular velocity as a new variable. Afterwards, follow the analogy with the Duffing oscillator. That is,

$$\frac{d\alpha}{dt} = \omega \quad \frac{d\omega}{dt} = \frac{M_{\text{net}}}{\Theta}$$

The net torque,  $M_{\text{net}}$ , has to change step by step during the solution; see Table 1. To exploit the power of the DS, dimensionless independent variables should be used. The dimensionless fundamental equation will be ready for computer simulation,

**Table 1.** Proposed topics, activities, number, and age of students.

Uniform static plus rotating magnetic fields	Video Dynamics Solver Chaotic attractor	Project work	Optional 2-3 students	18 years
Uniform static plus rotating magnetic fields	Dynamics Solver Stroboscopic projection Chaotic motion	Project work	Optional 3-5 students	
Uniform rotating magnetic field	Dynamics Solver Phase-plane representation from different reference frames	Project work	Optional 3-10 students	
Uniform static magnetic field	Dynamics Solver Phase-plane Representation of harmonic and damped oscillations	Project work	Optional 5-15 students	17 years
Uniform static magnetic field	Realization of a Helmholtz-coil Harmonic and damped oscillations Analogies pendulum - compass needle	Class activity	Mandatory 2 x 15-18 students	

**Table 2.** Compass needle: fundamental equations and topics.

	Equation	Topic
Fundamental equation (FE)	$\Sigma M = \Theta \beta$	Newton's law of motion
Conservative linear FE	$-\mu B_1 \alpha = \Theta_c \beta$	Harmonic oscillator
Conservative nonlinear FE	$-\mu B_1 \sin \alpha = \Theta_c \beta$	Nonlinear oscillator
Conservative driven FE	$-\mu B_1 \sin \alpha - \mu B_0 \sin(\alpha - \Omega t) = \Theta_c \beta$	Driven nonlinear oscillator
Dissipative damped FE	$-\gamma \omega - \mu B_1 \sin \alpha - \mu B_0 \sin(\alpha - \Omega t) = \Theta_c \beta$	Damped, driven, nonlinear oscillator
Independent variables	$t$ , time	
	$\alpha$ , angular displacement; $\omega$ , angular velocity; $\beta$ , angular acceleration; $\varphi$ , phase angle of the driving field	
Dependent variables	$d\alpha/dt = \omega$ ; $d\omega/dt = \beta$ or $d^2\alpha/dt^2 = \beta$	Derivatives with respect to time
Relations		
Parameters	$\Theta_c$ , $\mu$ , $B_0$ , $B_1$ , $\gamma$ , $\Omega$	

moreover, we will use the relations between dimensionless and real parameter values (see Table 3).

At the high school level, as was mentioned earlier, we suggest that the teacher formulate the dimensionless fundamental equation in a presentation and students will learn from this the meaning of dimensionless parameters. For example, A is proportional to dissipation, B is proportional to static magnetic field intensity, and C is proportional to rotating magnetic field intensity. At the university level, if students already understand the concept of derivatives and they are able to apply the chain rule in calculus, they can formulate the dimensionless fundamental equation individually with the aid of the teacher.

#### 4.4. Investigations with DS

##### 4.4.1. Linear restoring torque

Going through the program specified in Table 2 and using the notation of Table 3, as a first step we substitute  $M_{\text{net}} = -B\alpha$  and in the damped case  $M_{\text{net}} = -B\alpha - \gamma\omega$ . These equations are analogous to those of a simple pendulum, and in the case of the undamped pendulum it can be solved at high school. The didactics of exploiting the analogy were detailed earlier [8]. The various representations of this motion can be plotted by DS. These are very similar to those obtained during the investigation of the Duffing oscillator, so they shall not be presented.

##### 4.4.2. Nonlinear conservative and damped oscillators

Taking into account the nonlinearity of the net torque, we use  $M_{\text{net}} = -B \sin \alpha$  or in the damped case  $M_{\text{net}} = -B \sin \alpha - \gamma\omega$ . Here we can discuss the role of the dissipation in the classroom. The solutions obtained by DS allow comparison of the different

representation-types of conservative and non-conservative motions to the damped case; moreover, the linear case to the nonlinear one (Figs. 11 and 12).

Investigation of the angular velocity-angle phase-plane representations of the motion yield very interesting results. Applying  $M_{\text{net}} = -B\alpha$  and fixing the value of parameter B (Figs. 11 and 12) and changing only initial conditions of the motion, each curve will represent one conservative motion of a constant energy. Analysing these motions, students can associate the motion type with the energy.

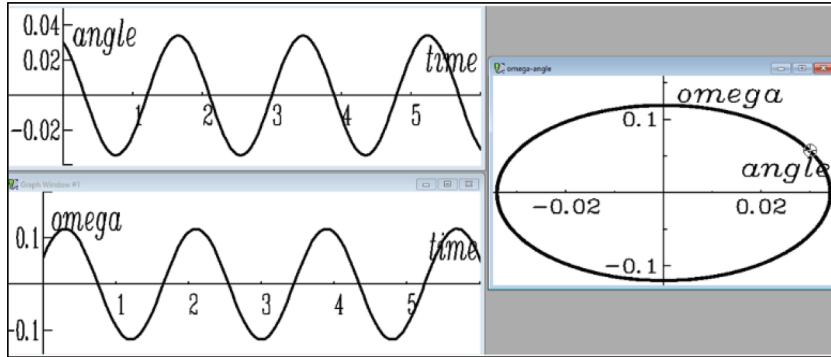
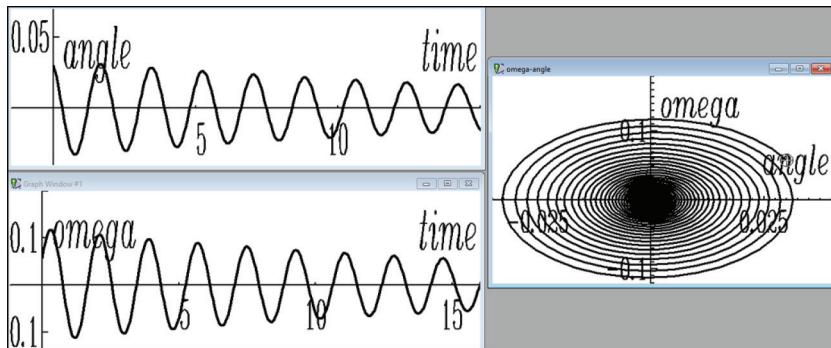
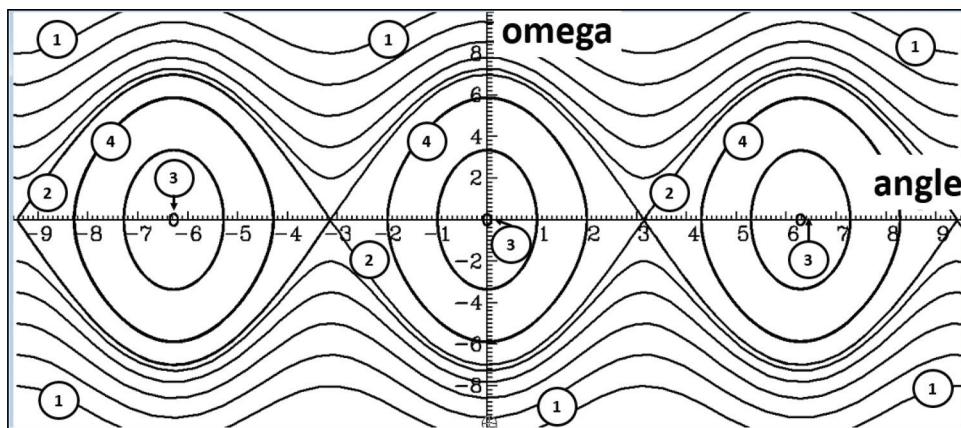
In Fig. 13 the curves of zones ③ and ④ represent oscillations, curve ② separates this region from zone ①. In the case of curve ②, the compass potential energy equals that of the unstable equilibrium position. The needle goes through the unstable equilibrium position with the dimensionless angular velocity  $2\sqrt{B}$ . Here  $\sqrt{B}$  is the proper angular frequency. If initial conditions determine a higher energy, in absolute terms, the compass motion will be rotational around the stable equilibrium positions. See periodic and not closed curves of zone ①.

If initial conditions determine a deeper energy, in absolute terms, the compass motion will be oscillatory. See the periodic and closed curves of zones ③ and ④. Anharmonic oscillations are represented by ellipses of zone ④. The curves of zone ③ can be considered as circles and represent harmonic oscillations.

Initial conditions of the curves: zone ① ( $\pm 3\pi, \pm 8$ ), ( $\pm 3\pi, \pm 6.5$ ), ( $\pm 3\pi, \pm 5$ ), ( $\pm 3\pi, \pm 3.5$ ), ( $\pm 3\pi, \pm 2$ ); zone ② ( $\pm 3\pi, \pm 2$ ); zone ③ ( $5(2\pi/360), 0$ ), ( $5(2\pi/360) \pm 2\pi, 0$ ); zone ④ (1, 0), (2, 0), ( $\pm 2\pi + 1, 0$ ), ( $\pm 2\pi + 2, 0$ ) and  $\sqrt{B} = 3.5$  is the dimensionless proper angular frequency of the system.

**Table 3.** Compass-needle: dimensionless variables, dimensionless fundamental equation.

Dimensionless independent variables	$\varphi = \Omega t$ , phase; $\alpha$ , angle (in radians)
Dimensionless dependent variables	$d\alpha/d\varphi = \omega/\Omega$ ; $d^2\alpha/d\varphi^2 = \beta/\Omega^2$
Dimensionless FE	$-A(d\alpha/d\varphi) - B \sin \alpha - C \sin(\alpha - \varphi) = d^2\alpha/d\varphi^2$
Dimensionless parameters	$A = \gamma/\Theta_c \Omega^2$ ; $B = \mu B_1/\Theta_c \Omega^2$ ; $C = \mu B_0/\Theta_c \Omega^2$

**Fig. 11.** Three representations of nonlinear conservative oscillator ( $A = 0.25$ ,  $B = 12.25$ ,  $\alpha_0 = 0.01$ , and  $\omega_0 = 0.058$ ). [Colour online.]**Fig. 12.** Three representations of nonlinear damped oscillator ( $A = 0.25$ ,  $B = 12.25$ ,  $\alpha_0 = 0.01$ , and  $\omega_0 = 0.058$ ). [Colour online.]**Fig. 13.** Compass needle in uniform magnetic field: phase-plane representation; each curve represents one motion.

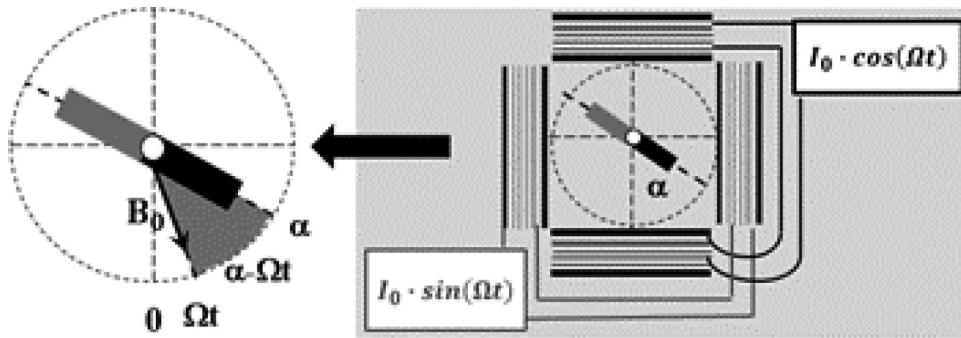
#### 4.4.3. Driven oscillators

##### 4.4.3.1. Conservative driven oscillators in a rotating magnetic field

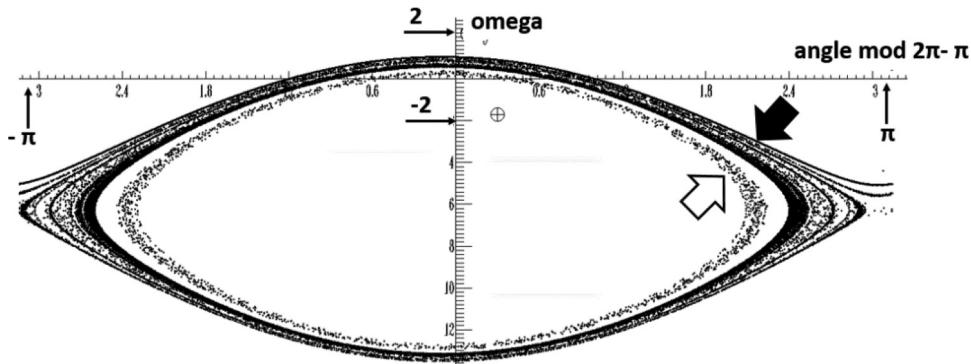
In the following we can describe the motion of a compass needle placed in the uniform rotating magnetic field of two identical perpendicular Helmholtz coil pairs, supplied by equal alternating electric currents, dephased by a quarter of a period. The superpo-

sition of the magnetic fields of the two sets of coils results in a uniform rotating field, which turns with the angular velocity  $\Omega$  of the currents. To formulate the fundamental equation from a referential frame, we use the angle between the actual position of the compass needle and the direction of rotating magnetic induction  $B_0$  (Fig. 14, left-hand panel),  $-\mu B_0 \sin(\alpha - \Omega t) = \Theta_c \beta$ , or in dimensionless form  $-C \sin(\alpha - \varphi) = d^2\alpha/d\varphi^2$ .

**Fig. 14.** Motion of a compass needle in rotating magnetic field of two perpendicular Helmholtz coils.



**Fig. 15.** Conservatively driven compass needle: transient and stable states ( $B = 0$ ,  $C = 13.0714$ ,  $x_0 = 0.3$ , and  $\omega_0 = -1.7$ ).



If we fix the reference frame to the rotating field, replacing the term  $(x - \Omega t)$  by  $\alpha'$ , we receive the same type of fundamental equation as in the uniform static case,  $-\mu B_0 \sin \alpha' = \Theta_c \beta'$ , or in dimensionless form  $-C \sin \alpha' = d^2 \alpha'/d\varphi^2$ . It can be seen that the equation agrees with that describing the motion in the laboratory system due to uniform static field. It is very important to recognise that the inertial forces are radial, and therefore their torques are zero.

The relation  $\alpha' = \alpha - \Omega t$  between angles from rotating and laboratory reference frames results in the relation between angular velocities:  $\omega' = \omega - \Omega$  and in that angular acceleration is the same, from both the referential frames:  $\beta = \beta'$ . For the analysis of the rotating field case one can use a rotational  $(\alpha', \omega')$  or a mixed phase plane  $(\alpha', \omega)$ . If we represent the motions, determined by the same initial conditions as in Fig. 13, from a rotational reference frame the representation is the equivalent of the uniform case representation. In the case of a mixed representation, the only difference will be a vertical shift by  $\Omega$ , according to the relation between angular velocities from different reference frames:  $\omega = \omega' + \Omega$ .

#### 4.4.3.2. Motion in superposed uniform static and rotating magnetic fields

In a conservative driven compass needle case, in the simultaneous presence of a uniform rotating and uniform static magnetic field, we worked with the conservative driven fundamental equation. For simulation, we used its dimensionless form:  $-B \sin \alpha - C \sin(\alpha - \varphi) = d^2 \alpha/d\varphi^2$ . In this case, a laboratory reference frame was used.

As we have a periodic driving force, it is reasonable to use stroboscopic projection, even if we see nothing special in the case of some parameter values. But with other parameter values, after a transient state, the system will follow the driving field. See the transient state between the white and black arrows in Fig. 15. The black arrow indicates stable rotation of the compass needle.

With parameter values of Fig. 16, we found two “oscillation-centers”. We can say that we have a system with two resonances. In other words, the compass needle oscillates unpredictably and occasionally around the uniform static or the uniform rotating magnetic field.

This results in a chaotic trajectory, which can be influenced by the intensities of magnetic fields. Graphically, it means that the zones of oscillation of uniform static and of uniform rotating field (zones ③ and zone ④) overlap [9].

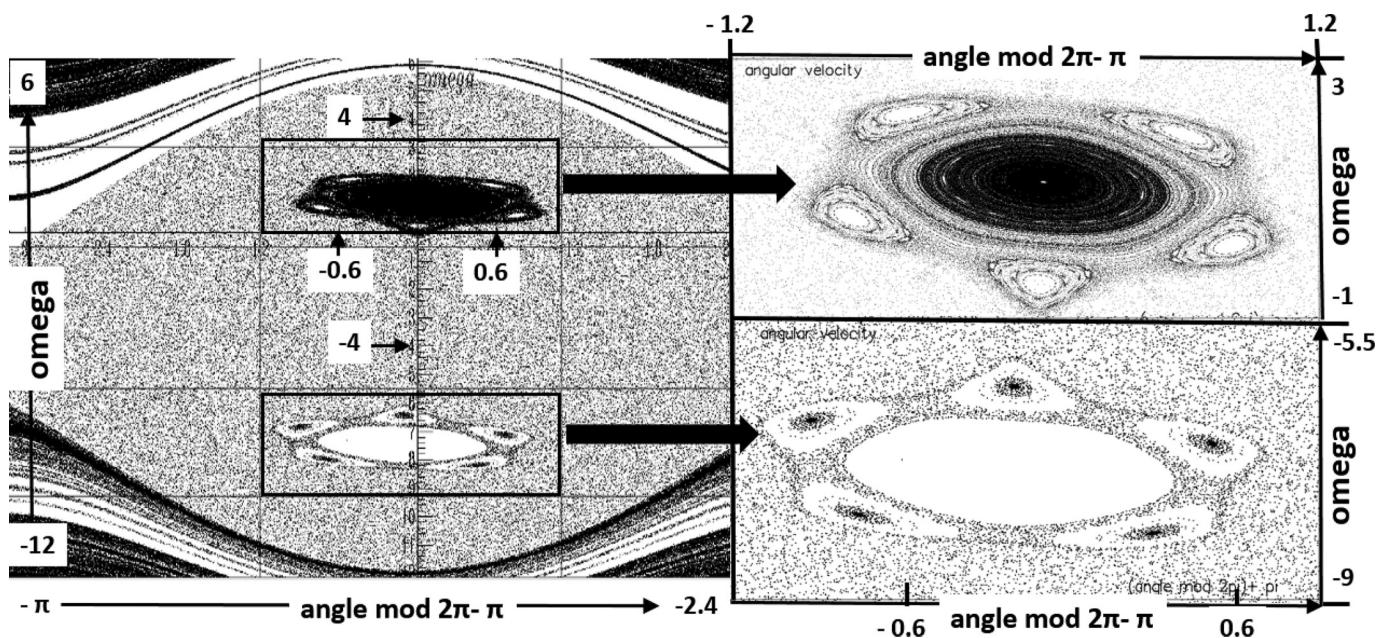
#### 4.4.3.3. Damped driven oscillators

The chaotic motions are characterised by sensitivity to initial conditions, at the same time their stroboscopic projections of any initial conditions can result in a fractal-structure subset. This subset of the stroboscopic projection is called chaotic attractor (Fig. 17). In a damped-driven compass needle, the motion can become chaotic with certain parameter values. In the following we try to find these parameter values, so that the chaotic attractor appears. We worked with the dissipative damped fundamental equation, and for the simulation we used its dimensionless fundamental equation.

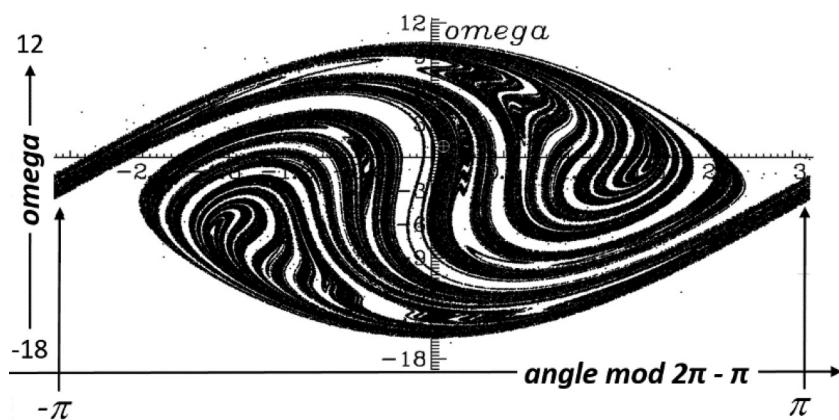
On stroboscopic projections in Figs. 18–20 parameter values  $A$ ,  $B$ ,  $C$  were varied, fixing initial conditions as  $\alpha_0 = 0.1$  and  $\omega_0 = 1$ . On the abscissa axis, the angle  $\alpha$  is between  $-\pi$  and  $\pi$ , on the ordinate axis the angular velocity  $\omega$  is between  $-15$  and  $20$ . To analyse the effect of energy dissipation we varied parameter  $A$ , fixing  $B$  and  $C$  values. We see chaotic attractors on these stroboscopic projections, which indicate chaotic motions at these parameter values. Increasing the dissipation, the extension of chaotic attractor in the phase-plane is reduced (Fig. 18).

To analyse the effect of uniform magnetic field, we varied the parameter  $B$  value between  $0$  and  $70$ , fixing dissipation and rotating field intensity as in Fig. 19. On the interval  $0 < B < 7.67$  we have no attractors on the stroboscopic projection, here the rotating

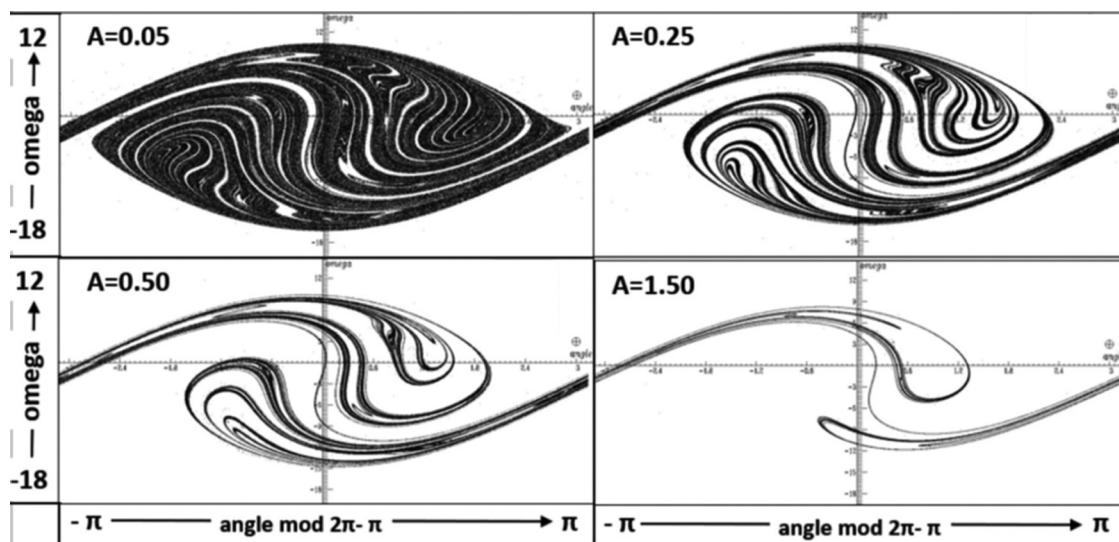
**Fig. 16.** Conservatively driven compass needle: a system with two resonances ( $B = 6.21593$  and  $C = 6.28318$ ).



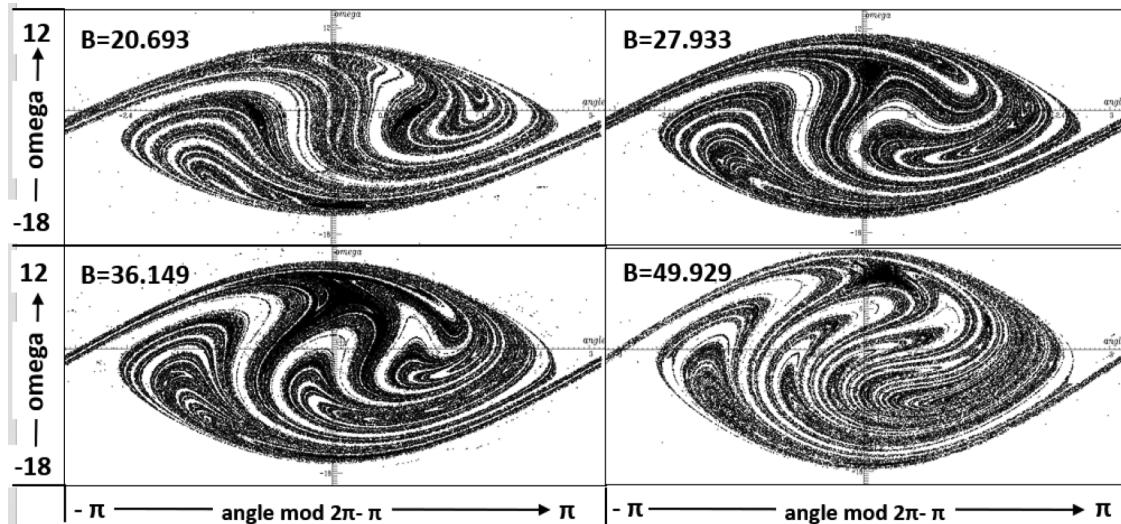
**Fig. 17.** Chaotic attractor of a damped-driven compass needle ( $A = 0.115$ ,  $B = 21$ , and  $C = 19$ ).



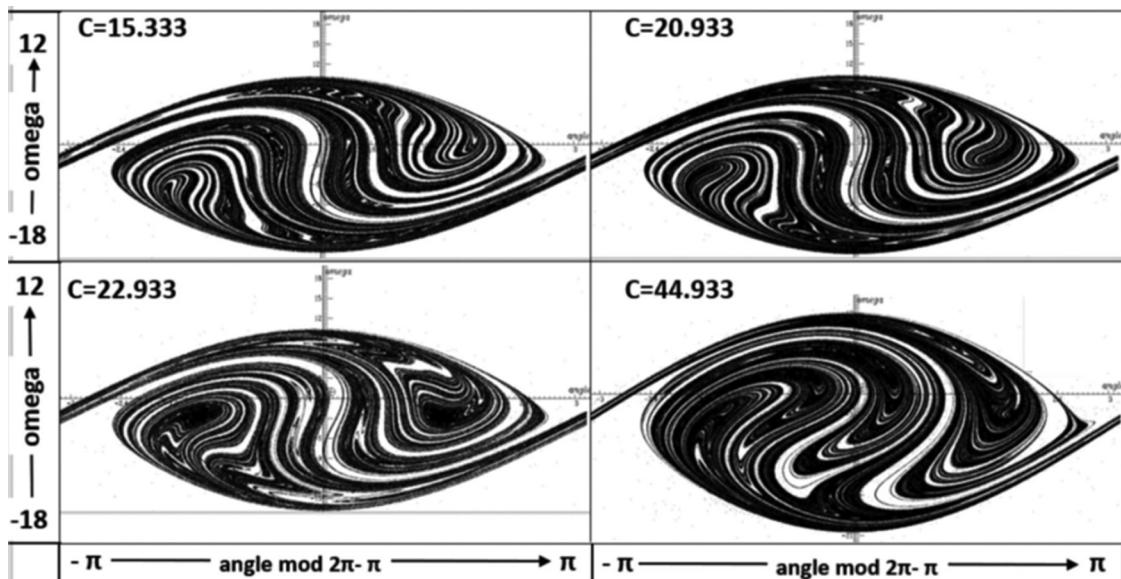
**Fig. 18.** Effect of dissipation on chaotic attractor ( $B = 21$  and  $C = 19$ ).



**Fig. 19.** Chaotic attractors and different uniform non-rotating fields ( $A = 0.1$  and  $C = 15$ ).



**Fig. 20.** Chaotic attractors and different rotating fields ( $A = 0.1$  and  $B = 20$ ).



field is dominant. On the interval  $7.67 \leq B \leq 50.25$  chaotic attractors appear four times; Fig. 19 shows four chaotic attractors of this interval, where  $B = 20.8, 27.9, 36.15$ , and  $49.93$ . The uniform and non-rotating field will be dominant if  $50.25 < B$ ; here, chaotic attractors disappeared. Increasing the uniform static field intensity from  $B = 20.8$  to  $49.93$ , we received an increase in the angular velocity interval from  $[-9, 15]$  to  $[-15, 17.5]$ .

To analyse the effect of rotating magnetic field, we varied  $C$  between 0 and 70, fixing dissipation and uniform static field as in Fig. 20. On the interval  $0 < C < 15$  we have no attractors on the stroboscopic projection. Here the uniform static field is dominant. If  $58.4 < C$  the rotating magnetic field is dominant, and the attractors disappear. Figure 20 shows four chaotic attractors of this interval, where  $C = 15, 21, 22.9$ , and  $58.4$ . Increasing the rotating field from  $C = 15$  to  $45$ , we received the widening of the angular velocity interval from  $[-9, 15]$  to  $[-12.5, 20]$ .

With many simulations as in Figs. 18, 19, or 20, a video was realized, where each picture was a stroboscopic projection. On these projections  $A$  and  $B$  parameter values were fixed; only the

parameter  $C$  value was changed from 0 to 100 by steps of 1. Finding something special, the students zoomed in on the  $C$  value to have a more detailed parameter-map. See a detailed video in ref. 15.

## 5. Conclusion

Studying the mathematical model of hardly realizable systems with simulations has led to exciting explorations. Simulations have revealed that under certain circumstances, deterministic systems exhibit unpredictable behaviour. Exploiting the advantageous properties of DS, students can focus on understanding the equations of motion of the systems and they can concentrate on the concepts necessary to describe chaotic motions. In high school classes, we propose to apply the time-dependent representations of DS, underlining that the physics teacher needs to know the dimensionless fundamental equation of the examined motion. The use of the DS fosters a deeper understanding of phase-plane and stroboscopic projection in undergraduate physics courses, therefore it can help teachers in a valuable way.

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