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# Integrating Dirac Approach to Quantum Mechanics into Physics Teacher Education 

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#### Abstract

There is a consensus on the importance of teaching/learning quantum mechanics in teacher education, but the way to implement this can vary widely depending on the goals. We cannot avoid the current, traditional secondary school curriculum, which includes wave-particle duality, bound states, and so on in most countries. But it is also worth preparing students for the second quantum revolution, which is about quantum computations. Approaches based on two-state systems not only provide an alternative foundation but also an introduction into quantum computations. They are also beneficial because educational researchers have already explored the power of two-state-system-based approaches such as the Dirac polarization approach in secondary school. So, applying two-state approaches can bring quantum computation closer, prepare the general formalism of quantum mechanics through illustrative examples and provide a new pedagogical insight in secondary school. In this paper proposals are presented for integrating the Dirac polarization approach into teacher education. It is shown how the formalism of QM can be adapted to the students' prior knowledge, and extra topics can be added bridging the gap between secondary school and university.


## INTRODUCTION

According to the literature, there are different options for introducing quantum mechanics ( QM ) based on two-state systems [1-3], one of the earliest uses the Dirac polarization approach [4] for this purpose [5]. The Dirac polarization approach in secondary school teaching is widely known by now, several papers present educational materials [5-9] and we can also be convinced that this teaching proposal is appropriate for secondary school level [10-17]. In addition, an international research groups on teaching/learning QM has also been formed [18]. However, the situation in the teacher education is different (as mentioned in [19]) because the prior knowledge of university students is broader, and the purposes may also be different: teachers need to know not only the foundations and formalism of quantum mechanics [20-22] but also guidelines of teaching/learning.

Prospective teachers meet with QM in their $4^{\text {th }}$ year of university studies, at Eötvös Loránd University (ELTE), Budapest, Hungary. In the first 3 years, students learn the basics of linear algebra, differential equations, complex numbers, probability theory and some atomic physics. This also means that we can make use of students' prior knowledge without, for example, avoiding the use of matrices as we do in secondary school. Taking these into account, we need to expand the curriculum, change some topics and better prepare the general formalism of QM. In this paper a pilot project is presented aiming the incorporation of the Dirac approach in physics teacher education.

## THE PHENOMENOLOGY OF DIRAC POLARIZATION APPROACH

The original secondary school curriculum can be found in more detail in the published literature [5-9]. Polarizationrelated experiments are easy to bring into the education of prospective teachers because polarization [23] is part of the Physics curriculum. Dirac approach applies polarization [24-26] to explore some of the fundamentals and formalism of QM. In the first part of the pilot project, university students explore the phenomena of polarization (Fig. 1.) and birefringent crystals, just like school kids but in a much shorter time. After that, students measure Malus' law, which
states that if polarized light falls on a polaroid, the intensity $I_{\mathrm{T}}$ of the transmitted light is $I_{\mathrm{T}}=I_{0} \cdot \cos ^{2} \theta$, where $I_{0}$ is the intensity of the incident polarized light and $\theta$ is the angle between the polarization direction of the polaroid and the polarization direction of the incident light.


FIGURE 1. Schematic diagram of polarization. Unpolarized light becomes polarized as it passes through the first polaroid. Then this polarized light passes through a second polaroid and gets reduced in intensity and obtains a different polarization property.
The light intensity is indicated with the number of lines, and the polarization property is indicated with the angle of the lines.
This classical phenomenon is converted into an interpretation wearing certain basic aspect of QM by introducing the photon hypothesis. By accepting that light consists of photons and the number of photons is proportional to the intensity of light (if the light is monochromatic), then Malus' law also holds for the number of photons $N$ : $N_{\mathrm{T}}=N_{0} \cdot \cos ^{2} \theta$. But if only one photon falls on a polaroid, Malus' law contradicts the indivisibility of photons. This problem can only be solved by assuming that Malus' law has a probabilistic meaning as shown in Fig. 2., i.e. a single photon is transmitted with probability $\operatorname{Pr}=\cos ^{2} \theta$.


FIGURE 2. The probabilistic meaning of Malus' law. If a photon is polarized at $45^{\circ}$ to the horizontal, then Malus' law is expressed as the probability of transmission of the photon through the second polaroid: $\operatorname{Pr}=\cos ^{2} \theta=1 / 2$.

The aim of the project is to build a framework for introducing preschool teacher students into QM by utilizing their knowledge of mathematics via simplified, model of QM. The pilot project lead by the author of this paper consisted of 3 times 90 -minute lessons in the first weeks of the semester, attended by 13 students as detailed in Table 1. In the first lesson, students carried out experiments and learned about Malus' law. In the second lesson, students performed statistical calculations based on the probabilistic interpretation of Malus' law. Finally, students were introduced to a simplified version (e.g., the use of complex numbers is avoided) of the full quantum formalism.

TABLE 1. The Dirac approach implemented in teacher education.

| $\begin{aligned} & \hline \text { \# Lessons } \\ & (/ 90 \mathrm{~min}) \\ & \hline \end{aligned}$ | Topics | Goals |
| :---: | :---: | :---: |
| \#1 | Exploring polarization | - Phenomenon of polarization and birefringent crystals. <br> - Malus' law. |
| \#2 | Calculations | - The validity of Malus' law for single photons. <br> - Calculate deviations and expectation values using the probabilistic interpretation of Malus' law. <br> - The uncertainty principle. |
| \#3 | Simplified formalism of QM | - The quantum state and transition probabilities. <br> - The superposition principle. <br> - Physical quantities represented with operators as linear combinations of projectors. Matrix form of operators. <br> - Commutator of operators and the uncertainty principle. <br> - The eigenvalue equation of photons passing through a polaroid. |

## ELEMENTARY STATISTICAL CALCULATIONS WITHOUT THE FORMALISM OF QM

This section presents calculations which are not part of the original secondary school curriculum [5-9] (but can be incorporated) and have been used in the pilot project for physics teacher education at ELTE. The Dirac approach allows statistical calculations without any quantum formalism, only using the probabilistic interpretation of Malus' law. These elementary statistics calculations provide evident and illustrative examples, which support later studies when students are faced with abstract questions. In this paper the general calculations are shown, but particular examples (with different direction of polaroids) were also given in the teaching units.

Consider a polarized photon falling on a polaroid. The probability of transmission is given by Malus' law $\operatorname{Pr}=\cos ^{2} \theta=p$, where $\theta$ is the angle between the polarization direction of light and the polarization direction of polaroid. If we know the probability of an event, we can carry out statistical calculations such as expectation value and standard deviation. First, we must recognize that the measurement has two possible outcomes, since the survival of single photon is a two-state problem. Similarly, to coin flipping, we can assign numbers to the permitted outcomes. We have chosen to measure value $\lambda_{1}=+1$ if the photon passes through the polaroid and value $\lambda_{2}=0$ if the photon is absorbed. The assignment can be arbitrary, but our choice appears to be natural in view of the physical interpretation of photon-polaroid interaction. The probability of measuring $\lambda_{1}$ is $p=\cos ^{2} \theta$, while the probability of measuring $\lambda_{2}$ is $1-p=\sin ^{2} \theta$. If we know the possible measured values and their probabilities, we can calculate the expectation value of a polarization measurement $A$ with a polaroid:

$$
\langle A\rangle=p \cdot \lambda_{1}+(1-p) \cdot \lambda_{2}=\cos ^{2} \theta \cdot(+1)+\sin ^{2} \theta \cdot 0=\cos ^{2} \theta
$$

The expectation value of quantity $A^{2}$ :

$$
\left\langle A^{2}\right\rangle=p \cdot \lambda_{1}^{2}+(1-p) \cdot \lambda_{2}^{2}=\cos ^{2} \theta
$$

With these, the standard deviation $\Delta A$ can be easily calculated:

$$
\begin{gathered}
(\Delta A)^{2}=\left\langle A^{2}\right\rangle-\langle A\rangle^{2}=\cos ^{2} \theta-\cos ^{4} \theta=\cos ^{2} \theta \sin ^{2} \theta=1 / 4 \sin ^{2}(2 \theta), \\
\Delta A=1 / 2|\sin 2 \theta| .
\end{gathered}
$$

The same is true for any polarization measurement with a polaroid with any orientation (and also for calcite crystals). By assigning the symbol $B$ to a polarization measurement with a polaroid of different orientation for the same ensemble of photons, the deviation is the following:

$$
\Delta B=1 / 2|\sin 2 \chi|
$$

where $\chi$ denotes the angle between the polaroid and the angle of polarization of the photons.
Notice that if the direction of polaroids $A$ and $B$ are not the same, neither are they perpendicular, then one of the deviations is always nonzero for a given polarization of incident photons. This is the consequence of the uncertainty principle: there are physical quantity pairs ( $A$ and $B$ in the example) that cannot accurately be measured simultaneously (in the same state), so one of the two quantities always has nonzero deviation as shown through a particular example in Fig. 3. This graphical interpretation of the uncertainty principle has not yet appeared in the literature.


FIGURE 3. The physical quantity $A$ means a measurement with a polaroid of polarization direction $\alpha=30^{\circ}$ (to the horizontal) on polarized photons whose state is given by the polarization angle $\varphi$ to the horizontal. The physical quantity $B$ also means a polarization measurement but with a polaroid of horizontal direction $\beta=0$ on the photonic state. The red curve is the graph of $1 / 2$ $|\sin 2 \theta|=1 / 2|\sin (2 \varphi-2 \alpha)|$ and the blue one is $1 / 2|\sin 2 \chi|=1 / 2|\sin (2 \varphi-2 \beta)|$. The graphs show that one of the quantities always deviates from the expectation value in any state. This is the consequence of the uncertainty principle. The circles mark states corresponding to the direction of one of the polaroids, when the deviation is zero. If one of the quantities can be precisely measured, the other one is uncertain as the dotted lines indicate.

Table 2 shows tasks from the pilot project related to a situation when horizontally polarized photons fall on a polaroid with $30^{\circ}$ polarization direction to the horizontal. The first column lists the tasks, and the second column shows the skills that students can acquire when solving the tasks.

TABLE 2. Elementary statistical calculations adjusted to the probabilistic interpretation of the Dirac approach.

| Task |  |
| :---: | :---: |
| a) What is the probability of passing through | Probabilistic features of QM (individual events |

the polaroid for a single photon?
b) What is the expectation value of the polarization measurement if the possible measurable outcomes are $\lambda_{1}=+1$ and $\lambda_{2}=0$. Mark the physical quantity by symbol $A$.
c) What is the deviation of the measurements?
d) What is the deviation of the polarization measurement with a polaroid of $45^{\circ}$ polarization direction (instead of $30^{\circ}$ ) in the same state? Mark the physical quantity by symbol $B$.
e) Sketch the deviations of quantity $A$ and $B$ on the same diagram! What can we say about the simultaneous deviations of quantities?

Statistical calculations without quantum formalism do not appear in the original secondary school curriculum, the uncertainty principle can be interpreted without the concept of standard deviations too. To this end the concept of "polarization property" has been defined in $[1,9]$, determined by a polarization measurement. A photon possesses a polarization property only if it has already passed through a polaroid with a certain direction or if we know for sure that it will pass through a polaroid with probability $100 \%$. In this interpretation, the uncertainty principle states that there are incompatible physical property pairs, where "incompatible property pairs" means that only one of the two properties can be assigned to a state [1, 9, 27]. This interpretation is consistent with the former elementary statistical description because if a photon possesses a polarization property, the standard deviation of the physical quantity assigned to this property is zero. So incompatible properties refer to two quantities which cannot be accurate (cannot be precisely measured) simultaneously because their deviations cannot be zero simultaneously.

## PREPARING THE FORMALISM OF QM

The school materials [5-9] introduces the quantum state as a real vector. It is this restriction because of which the formalism followed in the teaching material can be considered to represent only a model of QM. This setup encourages students to use vectors to represent the direction of polaroid, and afterwards the polarization state of photons.

Consider two polaroids with permitted directions $\boldsymbol{u}$ and $\boldsymbol{h}$ like e.g. in Fig. 2. Every photon prepared by the first polaroid has a well-defined quantum state ( $\boldsymbol{u}$ ) corresponding to the polarization direction of the polaroid. Photons with state $\boldsymbol{u}$ that already passed through the first polaroid fall onto the second polaroid of polarization direction $\boldsymbol{h}$. If they pass through the second polaroid, they acquire a new photonic state, namely $\boldsymbol{h}$. Thus, we can assign vectors to the photons after and before the transmission and can normalize them to unit vectors. Students can see that Malus' law appears as the square of the scalar product of these unit vectors:

$$
\operatorname{Pr}=\cos ^{2} \theta=\langle\boldsymbol{h}, \boldsymbol{u}\rangle^{2}=\left(\boldsymbol{h}^{\mathrm{T}} \cdot \boldsymbol{u}\right)^{2} .
$$

The secondary school teaching materials do not use the scalar product as matrix multiplication (the vector transposition is avoided), but at university we did. We marked the scalar product as $\langle\boldsymbol{h}, \boldsymbol{u}\rangle=\boldsymbol{h}^{\mathbf{T}} \cdot \boldsymbol{u}$ and explicitly indicated transposition because of a possible later use of complex vector spaces where transpose will be replaced with adjoint.

After exploring that the probability of transmission is the square of a scalar product, the polarization process can be considered as a transition between states. The probability $\operatorname{Pr}$ of a transmission through the second polaroid is equivalent with the state transition $\boldsymbol{u} \rightarrow \boldsymbol{h}$ of photons, i.e $\operatorname{Pr}(\boldsymbol{u} \rightarrow \boldsymbol{h})=\left(\boldsymbol{h}^{\mathrm{T}} \cdot \boldsymbol{u}\right)^{2}=p$ where $\boldsymbol{h}$ is the state after the measurement and $\boldsymbol{u}$ is the state before the measurement. We can assign a state vector $\boldsymbol{v}$ to the a photon which is going to be absorbed after passing through a polaroid: $\operatorname{Pr}(\boldsymbol{u} \rightarrow \boldsymbol{v})=\left(\boldsymbol{v}^{\mathrm{T}} \cdot \boldsymbol{u}\right)^{2}=1-p$ where the unit vector $\boldsymbol{v}$ is perpendicular to $\boldsymbol{h}$. So, if the incident photons are in state $\boldsymbol{v}$, then the probability of passing through a polaroid with direction $\boldsymbol{h}$ is $\operatorname{Pr}(\boldsymbol{v} \rightarrow \boldsymbol{h})=0$, that is consistent with a phenomenologically known property: if we take two polaroids with perpendicular directions in a row, there is no transmitted light.

The formal translation of the superposition principle is an immediate consequence of secondary school linear algebra: every vector can be represented as linear combinations of bases. In the case of polarization, the two bases can be the vectors assigned to transmission $(\boldsymbol{h})$ and absorption $(\boldsymbol{v})$ which we call eigenvectors; they point in the horizontal $(\boldsymbol{h})$ and vertical $(\boldsymbol{v})$, respectively. The state of an incident photon can be written as a linear combination of the eigenvectors: $\boldsymbol{u}=\psi_{1} \boldsymbol{h}+\psi_{2} \boldsymbol{v}$, where $\psi_{\mathrm{i}}$ are the weights or coefficients of the eigenvectors. So, the measuring device determines the eigenvectors, and this special linear combination can be called the superposition principle.

The coefficients $\psi_{1}$ and $\psi_{2}$ can be calculated as scalar products, just like in the general determination of the coefficients of eigenvectors in QM, namely $\psi_{\mathrm{i}}=\left\langle\boldsymbol{u}_{\mathrm{i}}, \boldsymbol{u}\right\rangle$ where $\boldsymbol{u}$ represents the state of photon and $\boldsymbol{u}_{\mathrm{i}}$ denotes one of the eigenvectors. Due to the probabilistic interpretation of transitions, $\psi_{1}^{2}+\psi_{2}^{2}=1$ holds in any two-state systems. It is worth mentioning that this calculation gives an opportunity to prepare students to realize that general states are superposition states, and the coefficient of eigenvectors should be determined with scalar products. Most of the students usually do not intend to take scalar products because the vectors are rarely unit vectors beyond the quantum world. Experience shows that students automatically calculate the coefficients by writing down a system of linear equations, in two-state system this implies only two equations. However, in infinite dimensional vector spaces (e.g., wave function) we cannot take infinitely many equations, so we have to calculate the coefficient with a different strategy. These evident examples provide opportunity to prepare the general formalism of QM.

The next step is to notice that the polaroid operates as a projector, rotating the state $\boldsymbol{u}$ of incident photons into one of the eigenvectors $\boldsymbol{h}$ and $\boldsymbol{v}$. The original teaching materials considers a polaroid of horizontal direction, and $\boldsymbol{h}$ and $\boldsymbol{v}$ denote horizontal and vertical states, respectively. It introduces the projectors with the formulas:

$$
\begin{aligned}
& \underline{\underline{P}}_{\mathrm{h}}=\boldsymbol{h} \boldsymbol{h} \\
& \underline{\underline{P}}_{\mathrm{v}}=\boldsymbol{v} \boldsymbol{v}
\end{aligned}
$$

These can be made acceptable in secondary schools by means of the following simple argument. The scalar product $\boldsymbol{h} \cdot \boldsymbol{u}$ gives the length of the projection of vector $\boldsymbol{u}$ in the direction of $\boldsymbol{h}$. The result of the operation $\underline{\underline{P}} \mathrm{~h}^{\boldsymbol{u}=\boldsymbol{h} \cdot(\boldsymbol{h} \cdot \boldsymbol{u})}$ is thus a vector parallel to $\boldsymbol{h}$ with the same length as the projection of $\boldsymbol{u}$ onto $\boldsymbol{h}$ due to the unit length of vectors. So, the projector $\underline{\underline{P}}_{\mathrm{h}}$ is an operation that projects vectors into the direction of $\boldsymbol{h}$, creating a new vector.

The secondary school curriculum [4-8] suggests that the operator of a polaroid (with horizontal direction) should be written as a linear combination of projectors:

$$
\underline{\underline{A}}=\lambda_{1} \underline{\underline{P}}_{\mathrm{h}}+\lambda_{2} \underline{\underline{P}}_{\mathrm{v}}
$$

The horizontal and vertical polarization measurements are preferable in secondary schools because they agree with the axes the ordinary coordinate systems.

At this point, the university level starts going beyond that of the secondary school material. Students faced with a more formal language (transpose of vectors and arbitrary orthogonal eigenvectors marked by $\boldsymbol{a}$ and $\boldsymbol{c}$ ) and wrote the projectors as:

$$
\begin{aligned}
& \underline{\underline{P}}_{\mathrm{a}}=\boldsymbol{a} \boldsymbol{a}^{\mathrm{T}} \\
& \underline{\underline{P}}_{\mathrm{c}}=\boldsymbol{\boldsymbol { c } ^ { \mathrm { T } }}
\end{aligned}
$$

which are called dyadic products in linear algebra [28]. Thus, in general we can write the superposition with arbitrary eigenvectors ( $\boldsymbol{a}$ and $\boldsymbol{c}$ ) as: $\boldsymbol{u}=\psi_{1} \boldsymbol{a}+\psi_{2} \boldsymbol{c}$. The eigenvectors are always determined by the measuring device. Writing the superposition with arbitrary eigenvectors was an extra part of the project, not included in the original secondary school level teaching materials. Furthermore, the operator of a measurement with an arbitrary polaroid can be given by rewriting the linear combination of the projectors by means of the dyads formed from $\boldsymbol{a}$ and $\boldsymbol{c}$ :

$$
\underline{\underline{A}}=\lambda_{1} \boldsymbol{a} \boldsymbol{a}^{\mathrm{T}}+\lambda_{2} \boldsymbol{c} \boldsymbol{c}^{\mathrm{T}} .
$$

This is the dyadic representation of a matrix in terms of its eigenvalues and eigenvectors, known for the students from their studies in linear algebra.

In the pilot project this illustrative way of writing the operator of a polaroid was further extended to representation of operators by means of real matrices. University students can explore that the operator of a polarization measurement is linear. They know from linear algebra that every linear transformation can be represented by a matrix. Matrices of polaroids multiply the length of non-zero eigenvectors by a scalar factor ( $\lambda_{1}=+1$ and $\lambda_{2}=0$ ), and general photonic states are projected into the eigenstates. Consequently, at the university we are able to create the matrices of polarisation measurements by assigning eigenvectors and eigenvalues to polaroids.

Let us mark with symbol $A$ the quantity assigned to a polarization measurement with a polaroid of polarization direction $\alpha$ to the horizontal. A polarization measurement sets the eigenvectors. Let $\boldsymbol{a}$ be the eigenvector corresponding to the transmission. It takes the form $\boldsymbol{a}=\binom{a_{l}}{a_{2}}=\binom{\cos \alpha}{\sin \alpha}$ in a horizontal-vertical frame. The other one, corresponding to absorption, is $\boldsymbol{c}=\binom{c_{1}}{c_{2}}=\binom{-\sin \alpha}{\cos \alpha}$. Thus, the matrix form of the operator $\underline{\underline{A}}$ of a polarization measurement taken with a polaroid of polarization direction $\alpha$ is the following:

$$
\underline{\underline{A}}=\lambda_{1} \boldsymbol{a} \boldsymbol{a}^{\mathrm{T}}+\lambda_{2} \boldsymbol{c} \boldsymbol{c}^{\mathrm{T}}=+1 \cdot\binom{a_{1}}{a_{2}} \cdot\left(a_{1}, a_{2}\right)+0 \cdot \boldsymbol{c} \boldsymbol{c}^{\mathrm{T}}=\left(\begin{array}{cc}
a_{1}^{2} & a_{1} a_{2} \\
a_{1} a_{2} & a_{2}^{2}
\end{array}\right)=\left(\begin{array}{cc}
\cos ^{2} \alpha & \cos \alpha \sin \alpha \\
\cos \alpha \sin \alpha & \sin ^{2} \alpha
\end{array}\right) .
$$

To my knowledge, this form has not been used in the literature dealing with the teaching of the Dirac approach. Notice that the matrix form of the operator is symmetric. Via this example, students see that every symmetric matrix has real eigenvalues and orthogonal eigenvectors, therefore a matrix form of an operator of a physical quantity should be symmetric. In later studies students can generalize this with self-adjoint operators which are the analogues of symmetric matrices in complex vector spaces.

Students can check that the eigenvalues of matrix $\underline{\underline{A}}$ are 1 and 0 , independent of $\alpha$, but the corresponding eigenvectors $\boldsymbol{a}$ and $\boldsymbol{c}$ do depend on $\alpha$. For an arbitrary state $\boldsymbol{u}=\binom{\cos \varphi}{\sin \varphi}$ of a photon with polarization direction $\varphi$ to the horizontal, and $\theta=\varphi-\alpha$ relative to the direction of the polaroid, the state of photons can be represented as the superposition of eigenvectors. Students can evaluate the scalar product $\psi_{1}=\langle\boldsymbol{a}, \boldsymbol{u}\rangle=\cos (\varphi-\alpha)=\cos \theta$ which is indeed the length of a unit vector of direction $\varphi$ projected onto the direction of $\alpha$. Students thus see that the transmission probability $\operatorname{Pr}(\boldsymbol{u} \rightarrow \boldsymbol{a})=\langle\boldsymbol{a}, \boldsymbol{u}\rangle^{2}=\psi_{1}^{2}$ and they can also check the validity of $\boldsymbol{u}=\psi_{1} \boldsymbol{a}+\psi_{2} \boldsymbol{c}$ via direct substitution of the constitutive terms.

Continuing the argumentation, let introduce another matrix marked with symbol $\underline{\underline{B}}$ assigned to another polaroid with polarization direction $\beta$ to the horizontal. With eigenvector $\boldsymbol{b}=\binom{b_{1}}{b_{2}}=\binom{\cos \beta}{\sin \beta}$ corresponding to the transmission, the matrix is the following:

$$
\underline{\underline{B}}=\left(\begin{array}{cc}
b_{1}^{2} & b_{1} b_{2} \\
b_{1} b_{2} & b_{2}^{2}
\end{array}\right) .
$$

Then the commutator $[\underline{\underline{A}}, \underline{\underline{B}}]=\underline{\underline{A}} \cdot \underline{\underline{B}}-\underline{\underline{B}} \cdot \underline{\underline{A}}$ is:

$$
[\underline{\underline{A}}, \underline{\underline{B}}]=\left(\begin{array}{cc}
0 & \left(a_{1}^{2}-a_{2}^{2}\right) b_{1} b_{2}-\left(b_{1}^{2}-b_{2}^{2}\right) a_{1} a_{2} \\
-\left(a_{1}^{2}-a_{2}^{2}\right) b_{1} b_{2}+\left(b_{1}^{2}-b_{2}^{2}\right) a_{1} a_{2} & 0
\end{array}\right) .
$$

Students thus see that the quantities $\underline{\underline{A}}$ and $\underline{\underline{B}}$ do not commute (if $\alpha \neq \beta$ or $|\alpha-\beta| \neq 90^{\circ}$ ). So the uncertainty principle is valid: these physical quantity pairs cannot be measured precisely simultaneously. This is the consequence of the fact that there are no common eigenvectors of the operators assigned to the physical quantities, thus one of the quantities always deviates from its expectation value which is a new view of the uncertainty principle via Dirac approach.

We can also measure the polarization of photons with birefringent calcite crystals instead of polaroids. In this case the two possible outcomes correspond to different trajectories. The measurable values with calcites can be chosen to be the eigenvalues $\lambda_{1}=+1$ and $\lambda_{2}=-1$. All previous calculations are similar but lead to different matrices because the eigenvalues are different even if the eigenvectors are the same. In the particular case when the polarization direction is horizontal or $45^{\circ}$ to the horizontal, the operators correspond to the real Pauli matrices. This choice is an advantageous one, as it fits to the Stern-Gerlach experiment. However, this article does not treat the cases of calcite crystals due to space limits, the literature [6-10] presents these, although without the matrix formalism.

Now let consider the effect of an operator to an arbitrary state as articles [4-8] suggest. Assume that the incident photons are in state $\boldsymbol{u}$, and calculate the effect of operator $\underline{\underline{A}}$ (related to quantity $A$ ) on state $\boldsymbol{u}$ in a two-state problem:

$$
\underline{\underline{A}} \boldsymbol{u}=\left(\lambda_{1} \boldsymbol{a} \boldsymbol{a}^{\mathrm{T}}+\lambda_{2} \boldsymbol{c} \boldsymbol{c}^{\mathrm{T}}\right) \cdot\left(\psi_{1} \boldsymbol{a}+\psi_{2} \boldsymbol{c}\right)=\lambda_{1} \psi_{1} \boldsymbol{a}+\lambda_{2} \psi_{2} \boldsymbol{c} .
$$

If this expression is multiplied by $\boldsymbol{u}^{\mathrm{T}}$ from the left, we get the expectation value of quantity $A$ in state $\boldsymbol{u}$ :

$$
\boldsymbol{u}^{\mathrm{T}} \underline{\underline{A}} \boldsymbol{u}=\left(\psi_{1} \boldsymbol{a}^{\mathrm{T}}+\psi_{2} \boldsymbol{c}^{\mathrm{T}}\right) \cdot\left(\lambda_{1} \psi_{1} \boldsymbol{a}+\lambda_{2} \psi_{2} \boldsymbol{c}\right)=\lambda_{1} \psi_{1}^{2}+\lambda_{2} \psi_{2}^{2}=\lambda_{1} p_{1}+\lambda_{2} p_{2}=\langle A\rangle_{\mathrm{u}}
$$

 expectation value of $A^{2}$ is also calculable:

$$
\left\langle A^{2}\right\rangle_{\mathrm{u}}=\boldsymbol{u}^{\mathrm{T}} \underline{\underline{A}}^{2} \boldsymbol{u}=\left(\psi_{1} \boldsymbol{a}^{\mathrm{T}}+\psi_{2} \boldsymbol{c}^{\mathrm{T}}\right) \underline{A}^{2}\left(\psi_{1} \boldsymbol{a}+\psi_{2} \boldsymbol{c}\right)=\left(\psi_{1} \boldsymbol{a}^{\mathrm{T}}+\psi_{2} \boldsymbol{c}^{\mathrm{T}}\right) \cdot\left(\lambda_{1}^{2} \psi_{1} \boldsymbol{a}+\lambda_{2}^{2} \psi_{2} \boldsymbol{c}\right)=\lambda_{1}^{2} \psi_{1}^{2}+\lambda_{2}^{2} \psi_{2}^{2}=\lambda_{1}^{2} p_{1}+\lambda_{2}^{2} p_{2}
$$

Note that we have obtained the general procedure for calculating the expectation values of a physical quantities in QM, which is the same as the result based on elementary statistics arguments presented earlier. In later studies, the generalization to arbitrary dimensions is much simpler. [5-9]

The matrix representation was used to revisit earlier results, too. In the particular case of a measurement with a polaroid of direction $\alpha$ to horizontal the students evaluated the expectation value by multiplying matrix $\underline{\underline{A}}$ with the state vector $\boldsymbol{u}=\binom{\cos \varphi}{\sin \varphi}$ from left and right:

$$
\langle A\rangle=(\cos \varphi, \sin \varphi)\left(\begin{array}{cc}
\cos ^{2} \alpha & \cos \alpha \sin \alpha \\
\cos \alpha \sin \alpha & \sin ^{2} \alpha
\end{array}\right)\binom{\cos \varphi}{\sin \varphi}=(\cos \alpha \cos \varphi+\sin \alpha \sin \varphi)^{2}=\cos ^{2} \theta
$$

This provides a new interpretation of Malus' law as the QM expectation value of measurements; thus the classical interpretation of Malus's law appears as the average number of transmitted photons (for an enormous amount of photons). Similarly, students obtained the expectation value of $A^{2}$ as $\cos ^{2} \theta$ with matrix multiplication. The standard deviation follows as: $\Delta A=1 / 2|\sin 2 \theta|$ which is the same as the result of the elementary statistical calculations.

Some students discovered that the above calculation of $\langle A\rangle$ is a bit complicated because vertical and horizontal bases were chosen, instead of the eigenvectors determined by the measuring instrument itself. The results appear in a much simpler form if we recall the diagonalization procedure of matrices, i.e. we choose the eigenvectors of the operator as the bases. In the coordinate frame of eigenvectors $\boldsymbol{a}$ and $\boldsymbol{c}$, the matrix takes a form containing the eigenvalues in the diagonal:

$$
\underline{\underline{A}}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)
$$

and the state vector of the incident photon with polarization direction $\varphi$ to the horizontal is just $\binom{\psi_{1}}{\psi_{2}}$. This is because $\boldsymbol{u}=\psi_{1} \boldsymbol{a}+\psi_{2} \boldsymbol{c}$ according to the superposition principle. In this frame, the evaluation of the standard deviation is also much simpler as:

$$
\langle A\rangle_{\mathrm{u}}=\left\langle A^{2}\right\rangle_{\mathrm{u}}=\left(\psi_{1}, \psi_{2}\right) \cdot\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \cdot\binom{\psi_{1}}{\psi_{2}}=\psi_{1}^{2}=\cos ^{2} \theta .
$$

This argument illustrates that physically relevant results do not depend on the choice of the reference frame.
Finally, by continuing the exploration of QM, students can accept that the argument holds for any two-state system beyond the polarization. Table 3 summarizes how the Dirac approach has been adapted to teacher education. The second column presents the secondary school approach, while the third column uses the early university language.

TABLE 3. Secondary school vs. early university level language. In this pilot project the language of mathematical expression is different from that of secondary schools because of the more advanced knowledge of university students. Innovations compared to the literature are highlighted in red.

| Topic | Secondary school level of <br> the Dirac approach | Early university level of <br> the Dirac approach |
| :---: | :--- | :---: |
|  | Complex numbers, matrices and also <br> the column vectors representation are <br> avoided. | Only real vectors and real matrices are <br> used in a way tuned to the general <br> formalism of QM. |
| Scalar product | $\boldsymbol{v} \cdot \boldsymbol{u}$ | $\langle\boldsymbol{v}, \boldsymbol{u}\rangle=\boldsymbol{v}^{\mathrm{T}} \cdot \boldsymbol{u}$ |

TABLE 3. Secondary school vs. early university level language. In this pilot project the language of mathematical expression is different from that of secondary schools because of the more advanced knowledge of university students. Innovations compared to the literature are highlighted in red. (cont.)

| Topic | Secondary school level of the Dirac approach | Early university level of the Dirac approach |
| :---: | :---: | :---: |
| Projector | $\underline{\underline{P}}^{\mathrm{v}}=\boldsymbol{v} \boldsymbol{v}$. | $\underline{\underline{P_{\mathrm{P}}}}=\boldsymbol{a} \boldsymbol{a}^{T}$ |
| Superposition | $\boldsymbol{u}=\psi_{1} \boldsymbol{h}+\psi_{2} \boldsymbol{v}$ <br> The bases are chosen to be the horizontal and vertical directions. | $\boldsymbol{u}=\psi_{1} \boldsymbol{a}+\psi_{2} \boldsymbol{c}$ <br> Arbitrary bases are chosen. |
| Operator | $\underline{\underline{A}}=\lambda_{1} \underline{\underline{P_{\mathrm{P}}}}+\lambda_{2} \underline{\underline{P}}_{\mathrm{v}}$ <br> The operators (polaroid of direction horizontal/vertical) as dyads avoiding the matrix representation. <br> The eigenvalues are the possible measurable outcomes.] If the instrument is a polaroid, the choice of +1 and 0 is relevant. <br> However, in the case of calcite crystals the choice $\pm 1$ is more suitable. | $\begin{gathered} \underline{\underline{A}=\lambda_{1} \boldsymbol{a} \boldsymbol{a}^{\mathrm{T}}+\lambda_{2} \boldsymbol{c} \boldsymbol{c}^{\mathrm{T}}=} \\ =\left(\begin{array}{cc} \lambda_{1} a_{1}^{2}+\lambda_{2} c_{1}^{2} & \lambda_{1} a_{1} a_{2}+\lambda_{2} c_{1} c_{2} \\ \lambda_{1} a_{1} a_{2}+\lambda_{2} c_{1} c_{2} & \lambda_{1} a_{2}^{2}+\lambda_{2} c_{2}^{2} \end{array}\right) . \end{gathered}$ <br> The direction of polaroid is arbitrary. <br> Eigenvectors and eigenvalues are determined by the measurement. <br> This matrix is symmetric because its eigenvalues are real, so it represents a physical quantity. <br> Diagonalization of matrices. |
| Expectation value | $\langle A\rangle_{\mathrm{u}}=p_{1} \cdot \lambda_{1}+p_{2} \lambda_{2}$ | $\rangle_{\mathrm{u}}=\boldsymbol{u}^{\mathrm{T}} \underline{\underline{A}} \boldsymbol{u}=\lambda_{1} \psi_{1}^{2}+\lambda_{2} \psi_{2}^{2}$ |
| Malus' law | $\begin{gathered} \operatorname{Pr}=\cos ^{2} \theta . \\ \operatorname{Pr}=(\boldsymbol{u} \rightarrow \boldsymbol{v})=(\boldsymbol{v} \cdot \boldsymbol{u})^{2} \end{gathered}$ | Malus' law is the expectation value of the measurement. |
| Variance | $(\Delta A)^{2}=$ | $\left\langle A^{2}\right\rangle-\langle A\rangle^{2}$ |
| Uncertainty principle | There are incompatible physical properties. <br> There are physical quantity pairs that cannot accurately be precisely measured simultaneously. So one of the quantities is uncertain. | If the eigenvectors of the operators assigned to the physical quantities are not the same, then one of the quantities always deviates from its expectation value. <br> The uncertainty principle holds when two operators of physical quantities do not commutate. |

Table 4 presents new tasks of the pilot project. In task 1 horizontally polarized photons fall on a polaroid with $30^{\circ}$ polarization direction to horizontal. In task 2 the operator $\underline{\underline{A}}$ of the measurement of polarization with the polaroid is known to be $\underline{\underline{A}}=1 / 2\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$.

TABLE 4. New tasks in relation to the Dirac approach given to university students.

| Task |  |  |
| :--- | :--- | :--- |
| 1/a) | Determine the eigenvectors of the polaroid <br> and write the photonic state as the |  |
|  | superposition of eigenvectors! | Using the bases determined by the measurement. <br> 1/b)What is the probability of transmission and <br> absorption using the superposition <br> principle? Why is it necessary to write the |
|  |  |  |
| state of photons as a superpositar products. |  |  |

1/c) Express the operator of a polaroid as a dyadic expression!
$1 / \mathrm{d})$ What is the matrix form of the operator?

Operators of physical quantities using dyads.
The physical and mathematical meaning of projectors and operators.
Real symmetric matrices represent operators.

TABLE 4. New tasks in relation to the Dirac approach given to university students (cont.)

| Task |  |  |  | Topic |
| :--- | :--- | :--- | :---: | :---: |
| 1/e) | Calculate the expectation value and the <br> standard deviation of physical quantity $A$ <br> corresponding to a polarization <br> measurement with the polaroid by means <br> of the general formalism of QM! | Preparing the general formalism of QM. |  |  |

## CONCLUSIONS

One of the purposes of this article is to show that the introduction of Dirac approach is efficient and useful at the university level: students become acquainted with simple examples illustrating the abstract formalism of QM, and as prospective teachers they can also acquire a teachable method of secondary school level QM.

New tasks are formulated, the language of the teaching material is brought on a higher level of mathematics, and some proposals are made. The teaching material contains secondary school level elementary statistical calculations not yet published earlier, the real vector formalism extended by column vectors, transpose of vectors, and the representation of operators by symmetric matrices. The paper presents a possible interpretation of Malus' law as the expectation value of polarization measurements. This article also shows a new approach to the uncertainty principle presenting it with simultaneous deviations of quantity pairs and also with commutators. Furthermore, the pilot project uses arbitrary bases and diagonalization of matrices which supplement the case of horizontal/vertical bases treated in the school material. In the future, integrating complex numbers and quantum entanglement into the teaching material for students can be the next step. Perhaps these innovations will not only help prospective teachers to learn the formalism and foundations of QM, but also contribute to the methodology of teaching and the cultural knowledge of the QM way of thinking.

The students started the semester by refreshing the knowledge of statistic, linear algebra and exploring some of the laws of QM through the Dirac approach. At the end of the pilot project, students wrote a test related to the subject of this paper. The results were remarkably promising, students enjoyed the project. After the pilot project, students continued the quantum seminar with higher dimensional systems, and then the wave formalism was introduced by infinite dimensional vectors and matrices. As a result of the project, the next part of the semester was found easier by the students because they understood the concepts, foundations, and formalism through simple examples earlier. In particular, the interpretation of the energy spectrum and eigenfunctions of the Schrödinger equation was easier to understand after the two-state approach. I had the opportunity to point out that the coefficients $\psi_{1}$ and $\psi_{2}$ can be considered as a preimage of energy eigenfunctions $\psi(x)$, since high dimensional vectors can approximate continues functions very well. The project also provided new pedagogical skills and different approaches of phenomena. The pilot project has proved to be encouraging for a continuation in the future.

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